# CHAPTER 6 SOLUTION METHODS AND STABILITY 

# Very Rough Draft-use equations at your own peril 

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### 6.1 INTRODUCTION

This Chapter describes solution procedures for nonlinear finite element discretizations. In addition, methods for examining the physical stability of solutions and the stability of solution procedures are described.

The first part of the chapter describes time integration, the procedures used for integrating the discrete momemtum equation and other time dependent equations in the system, such as the constitutive equation. We begin with the simplest of methods, the central difference method for explicit time integration. Next the family of Newmark $\beta$-methods, which encompass both explicit and implicit methods, are described. Explicit and implicit methods are compared and their relative advantages described. As part of implicit methods, the solution of equilibrium equations is also examined.

A critical step in the solution of implicit systems and equilibrium problems is the linearization of the governing equations. Linearization procedures for the equations of motion, and as a special case, the equilibrium equations are described.

### 6.2 EXPLICIT METHODS

In this Section the major features of explicit and implicit time integration methods for the discretized momentum equation and solution methods for the discrete equilibrium equations are described. The methods are described in the context of Lagrangian meshes, but can be extended to Eulerian and ALE meshes with some techniques described in Chapter 7. The description of the solution procedures of equilibrium problems is combined with the description of implicit procedures for dynamic problems, because, as we show later, the methodologies are almost identical; the solution of a static problem by an implicit method only requires that the inertial term be dropped.

To illustrate the major features of explicit and implicit methods for time integration, the solution of the equations of motion is first considered for rateindependent materials. In this class of equations, we can avoid some of the complications that arise in the treatment of rate-dependent materials but still illustrate the most important properties of explicit and implicit methods. We will first describe explicit and implicit methods using only a single time integration formula: the central difference method for explicit time integration and the

Newmark $\beta$-methods for implicit integration. In Section X, other time integration formulas are considered.
6.2.1. Central Difference Method. The central difference method is among the most popular of the explicit methods in computational mechanics and physics. It has already been discussed in Chapter 2, where it was chosen to demonstrate some nonlinear solutions in one dimension. The central difference method is developed from central difference formulas for the velocity and acceleration. We consider here its application to Lagrangian meshes with rateindependent materials. Geometric and material nonlinearites are included, and in fact have little effect on the time integration algorithm.

For the purpose of developing this and other time integrators we will use the following notation. Let the time of the simulation $0 \leq t \leq t_{E}$ be subdivided into time intervals, or time steps, $\Delta t^{n}, n=1$ to $n_{T S}$ where $n_{T S}$ is the number of time steps and $t_{E}$ is the end-time of the simulation; $\Delta t^{n}$ is also called the $n$th time increment. The variables at any time step are indicated by a superscript; thus $t^{n}$ is the time at time step $n, t^{0}=0$ is the beginning of the simulation and $\mathbf{d}^{n} \equiv \mathbf{d}\left(t^{n}\right)$ is the matrix of nodal displacements at time step $n$. Time increment $n$ is given by

$$
\begin{equation*}
\Delta t^{n}=t^{n}-t^{n-1} \quad \Delta t^{n+\frac{1}{2}}=\frac{1}{2}\left(\Delta t^{n}+\Delta t^{n+1}\right) \tag{6.2.1}
\end{equation*}
$$

where the second equation gives the midpoint time step.
The central difference formula for the velocity is

$$
\begin{equation*}
\dot{\mathbf{d}}^{n+\frac{1}{2}} \equiv \mathbf{v}^{n+\frac{1}{2}}=\frac{1}{\Delta t^{n+\frac{1}{2}}}\left(\mathbf{d}^{n+1}-\mathbf{d}^{n}\right), \quad \mathbf{d}^{n+1}=\mathbf{d}^{n}+\Delta t^{n+\frac{1}{2}} \mathbf{v}^{n+\frac{1}{2}} \tag{6.2.2}
\end{equation*}
$$

where the second equation gives the corresponding integration equation which is obtained by a rearrangement of the first. The acceleration is given by

$$
\begin{equation*}
\ddot{\mathbf{d}}^{n} \equiv \mathbf{a}^{n}=\frac{1}{\Delta t^{n}}\left(\mathbf{v}^{n+\frac{1}{2}}-\mathbf{v}^{n-\frac{1}{2}}\right) \quad \mathbf{v}^{n+\frac{1}{2}}=\mathbf{v}^{n-\frac{1}{2}}+\Delta t^{n} \mathbf{a}^{n} \tag{6.2.3a}
\end{equation*}
$$

As can be seen from the above, the velocities are defined at the midpoints of the time intervals, or at half-steps. By substituting (6.2.2a) and its counterpart for the previous time step into (6.2.3), the acceleration can be expressed directly in terms of the displacements

$$
\begin{equation*}
\ddot{\mathbf{d}}^{n} \equiv \mathbf{a}^{n}=\frac{\Delta t^{n-\frac{1}{2}}\left(\mathbf{d}^{n+1}-\mathbf{d}^{n}\right)-\Delta t^{n+\frac{1}{2}}\left(\mathbf{d}^{n}-\mathbf{d}^{n-1}\right)}{\Delta t^{n} \Delta t^{n-\frac{1}{2}} \Delta t^{n+\frac{1}{2}}} \tag{6.2.3b}
\end{equation*}
$$

For the case of equal time steps the above reduces to

$$
\begin{equation*}
\ddot{\mathbf{d}}^{n} \equiv \mathbf{a}^{n}=\frac{\left(\mathbf{d}^{n+1}-2 \mathbf{d}^{n}+\mathbf{d}^{n-1}\right)}{\left(\Delta t^{n}\right)^{2}} \tag{6.2.3c}
\end{equation*}
$$

This is the well known central difference formula for the second derivative of a function.

We now consider the time integration of the undamped equations of motion for rate-independent materials, Eq. (4.x.x.), which at time step $n$ are given by

$$
\begin{align*}
& \mathbf{M a} \mathbf{a}^{n}=\mathbf{f}^{n}=\mathbf{f}^{e x t}\left(\mathbf{d}^{n}, t^{n}\right)-\mathbf{f}^{i n t}\left(\mathbf{d}^{n}, t^{n}\right)  \tag{6.2.4a}\\
& \text { subject to } g_{I}\left(\mathbf{d}^{n}\right)=0, I=1 \text { to } n_{c} \tag{6.2.4b}
\end{align*}
$$

where ( 6.2 .4 b ) is a generalized representation of the $n_{c}$ displacement boundary conditions; constraints may also arise from other conditions on the model. The mass matrix in this expression is considered constant because as noted in Section X, it is time independent for a Lagrangian mesh. Methods for Eulerian meshes are discussed in Chapter 7. The internal and external nodal forces are functions of the nodal displacements and the time. The external loads are usually prescribed as functions of time; they may also be functions of the nodal displacements because they may depend on the configuration of the structure, as when pressures are applied to the surfaces which undergo large deformations. The dependence of the internal nodal forces on displacements is quite obvious: the nodal displacements determine the strains, which in turn determine the stresses and hence the nodal internal forces by Eq. (4.4.11). Internal nodal forces are generally not directly dependent on time, but there are situations of engineering relevance when this is the case; for example, when the temperature is prescribed as a function of time, the stresses and hence the internal nodal forces depend directly on time.

The equations for updating the nodal velocities and displacements are obtained as follows. Substituting Eq. (6.2.4a) into (6.2.3b) gives

$$
\begin{equation*}
\mathbf{v}^{n+\frac{1}{2}}=\Delta t^{n} \mathbf{M}^{-1} \mathbf{f}^{n}+\mathbf{v}^{n-\frac{1}{2}} \tag{6.2.5}
\end{equation*}
$$

which provides an update for the nodal velocities; the displacements are then updated by (6.2.2).

At any time step $n$, the displacements $\mathbf{d}^{n}$ will be known. The nodal forces $\mathbf{f}^{n}$ can be determined by using in sequence the strain-displacement equations, the constitutive equation and the relation for the nodal internal forces. Thus the entire right hand side of (6.2.5) can be evaluated, which gives $v^{n+1 / 2}$, and the displacements $\mathbf{d}^{n+1}$ at time step $n+1$ can be determined by (6.2.2b). The entire update can be accomplished without solving any system equations provided that the mass matrix $\mathbf{M}$ is diagonal. This is the salient characteristic of an explicit method:
in an explicit method, the time integration of the discrete momentum equations for a finite element model does not require the solution of any equations.

In numerical analysis, integration methods are classified according to the structure of the time difference equation. The difference equations for first and second derivatives are written in the general forms

$$
\begin{equation*}
\sum_{n=0}^{n_{S}}\left(\alpha_{n} \mathbf{d}^{n_{S^{-n}}}-\Delta t \beta_{n} \dot{\mathbf{d}}^{n}\right)=0 \quad \sum_{n=0}^{n_{S}}\left(\bar{\alpha}_{n} \mathbf{d}^{n_{S^{-n}}}-\Delta t^{2} \bar{\beta}_{n} \ddot{\mathbf{d}}^{n}\right)=0 \tag{6.2.6}
\end{equation*}
$$

where $n_{S}$ is the number of steps in the difference equation. The difference formula for the first or second derivatives is called explicit if $\beta_{0}=0$ or $\bar{\beta}_{0}=0$, respectively. From (6.2.3c) it can be seen that $\bar{\beta}_{0}=0, \bar{\beta}_{1}=1, \bar{\beta}_{2}=0$, so the formula is explicit. Thus the difference formula is called explicit if the equation for the function at time step $n$ only involves the derivatives at previous time steps. Difference equations which are explicit according to this classification generally lead to solution schemes which require no solution of equations. In most cases there is no benefit in using explicit schemes which involve the solution of equations, so the use of such explicit schemes is rare. There are a few exceptions. For example, if the consistent mass is used with the central difference method, even though the difference equation is classified as explicit, system equations still need to be solved in the update.
6.2.2. Implementation. A flow chart for explicit time integration of a finite element model with rate-independent materials is shown in Box 6.1. This flowchart generalizes the flowchart given in Chapter 2 by considering nonzero initial conditions, a variable time step and including elements which require more than one-point quadrature. The primary dependent variables in this flowchart are the velocities and the Cauchy stresses. Initial conditions must be given for the velocitites, the Cauchy stresses, and all state variables of the materials in the model. The displacements are initially considered to vanish.

Flowchart inorrect, half missing on time steps, not $n$ order

## Box 6.1

## Flowchart for Explicit Time Integration

1. Initial conditions and initialization: set $\mathbf{v}^{0}, \sigma^{0}$, and other material state variables;

$$
\mathbf{d}^{0}=0, n=0, t=0 ; \text { compute } \mathbf{M}
$$

2. $\operatorname{getf}\left(\mathbf{f}^{n}, \Delta t_{\text {crit }}\right)$
3. compute accelerations $\mathbf{a}^{n}=\mathbf{M}^{-1} \mathbf{f}^{n}$
4. compute kinetic energy and check energy balance, see Section ??
5. update nodal velocities: $\mathbf{v}^{n+\frac{1}{2}}=\mathbf{v}^{n}+\frac{1}{2} \Delta t^{n} \mathbf{a}^{n}$
6. enforce velocity boundary conditions:

$$
\text { if node } I \text { on } \Gamma_{v_{i}}: v_{i I} \frac{n+\frac{1}{2}}{2}=\bar{v}_{i}\left(\mathbf{x}_{I}, t_{n+\frac{1}{2}}\right)
$$

7. update nodal displacements: $\mathbf{d}^{n+1}=\mathbf{d}^{n}+\Delta t^{n+\frac{1}{2}} \mathbf{v}^{n+\frac{1}{2}}$
8. update counter and time: $n \leftarrow n+1, t \leftarrow t+\Delta t$
9. update nodal velocities: $\mathbf{v}^{n+1}=\mathbf{v}^{n+\frac{1}{2}}+\frac{1}{2} \Delta t^{n} \mathbf{a}^{n}$
10. output, if simulation not complete, go to 2

Subroutine $\operatorname{getf}\left(\mathbf{f}^{n}, \Delta t_{\text {crit }}\right)$
0 . initialization: $\mathbf{f}^{n}=0, \Delta t_{\text {crit }}=\infty$

1. compute external nodal forces $\mathbf{f}^{\text {ext,n}}$ which are global
2. loop over elements $e$
i. GATHER element nodal displacements and velocities
ii. $\mathbf{f}_{e}^{i n t, n}=0$
iii. loop over quadrature points $\xi_{Q}$
3. if $n=0$, go to 8
4. compute measures of deformation: $\mathbf{D}^{n-\frac{1}{2}}\left(\xi_{Q}\right), \mathbf{F}^{n}\left(\xi_{Q}\right), \mathbf{E}^{n}\left(\xi_{Q}\right)$
5. compute stress $\sigma^{n}\left(\xi_{Q}\right)$ by constitutive equation
6. $\mathbf{f}_{e}^{i n t, n} \leftarrow \mathbf{f}_{e}^{i n t, n}+\left.\mathcal{B}^{T} \sigma^{n} \bar{w}_{Q} J\right|_{\xi_{Q}}$

END quadrature point loop
iv. compute external nodal forces on element, $\mathbf{f}_{e}^{\text {ext }, n}$
v. $\mathbf{f}_{e}^{n}=\mathbf{f}_{e}^{e x t, n}-\mathbf{f}_{e}^{\mathrm{int}, n}$
vi. compute $\Delta t_{\text {crit }}^{e}$, if $\Delta t_{\text {crit }}^{e}<\Delta t_{\text {crit }}$ then $\Delta t_{\text {crit }}=\Delta t_{\text {crit }}^{e}$
vii. SCATTER $\mathbf{f}_{e}^{n}$ to global $\mathbf{f}^{n}$
3. END loop over elements

In this algorithm, the accelerations are first integrated to obtain the velocities. The integration of the velocities is broken into two half-steps so that the velocities are available at an integer step in the computation of the energy balance. The displacements are computed in each time step by integrating the velocities.

The main part of the procedure is the calculation of the nodal forces from the nodal displacements at a given time step, which is performed in getf. In this subroutine, the equations governing a continuum are used along with the gather/scatter procedures:

1. the nodal displacements of the element are extracted from the global matrix of nodal displacements by the "gather " operation;
2. the strain measures are computed at each quadrature point of the element;
3. the stresses are computed by the constitutive equation at each quadrature point;
4. the internal nodal forces are computed by integrating the product of the B matrix and the stresses over the domain of the element with the Cauchy stress;
5. the nodal forces of the element are scattered into the global array.

In the first time step, the strain measures and the stress are not computed. Instead, as shown in the flowchart, the initial stresses are used to obtain the internal nodal forces.

The flowchart shows the algorithm with the matrix form of the internal force computation, in which the stress tensor is stored as a square matrix and the $B$ matrix is used. The change to the Voigt form only requires the use of a column matrix for the stresses and the $\mathbf{B}$ matrix, (4.5.14). Similarly, the internal force computation can be changed to the total Lagrangian format by replacing the discrete values of the integrand in step 10 by the integrands of (B4.8.2).

Most essential boundary conditions are easily handled in explicit methods. For example, if the velocities or displacements are prescribed as functions of time along any boundary, then the velocity/displacement boundary conditions can be enforced by setting the nodal velocities according to the data:

$$
\begin{equation*}
v_{i I}^{n}=\bar{v}_{i}\left(\mathbf{x}_{I}, t^{n}\right) \tag{6.2.7}
\end{equation*}
$$

If the data is not available on the nodes, the least square procedure given in Section 2.4.5 can be used to fit the nodal values.

The velocity boundary conditions can also be enforced in local coordinate systems as shown in the Box 6.1. In that case, the equations of motion at these nodes must be expressed in the local coordinate system, so the nodal force components must be expressed in the local coordinate systems before assembly and time integration. The boundary condition is also enforced in the local coordinate system. The orientation of the local coordinate system may vary with time but the time integration formulas must then be modified to account for the additional terms in the equations of motion.

When essential boundary conditions are given as linear or nonlinear algebraic equations relating the displacements, the implementation is more complicated. One approach is to use a linearization of the constraint. Consider for example the nonlinear constraint

$$
\begin{equation*}
\mathbf{g}(\mathbf{d}(t))=0 \tag{6.2.8}
\end{equation*}
$$

where $\mathbf{g}(\mathbf{d}(t))$ is a linear or nonlinear algebraic function of the nodal displacements. If the constraint involves integral or differential relationships, such as a dependence on the velocities, it can be put in the above form by using
difference equations or a numerical approximation of the integral. The above can be linearized as follows:

$$
\begin{equation*}
\left(\frac{\partial \mathcal{G}\left(\mathbf{d}^{n}\right)}{\partial d_{a}}+\frac{\partial \mathcal{G}\left(\mathbf{d}^{n+1}\right)}{\partial d_{a}}\right) v_{a}^{n+1 / 2}=0 \tag{6.2.9}
\end{equation*}
$$

After a large number of time steps, linearizations such as the above combined with a central difference update of the displacements may substantially violate the constraint. This drift in the enforcement of the constraint can be avoided by correcting the linearized update so that the constraint is enforced exactly at the next time step, $n+1$. When accurate treatment of the constraints is important, techniques for differential-algebraic equations should be used, Petzold (??).

As can be seen from the flowchart, an explicit method is easily implemented. Furthermore, explicit time integration is very robust, by which we mean that the explicit procedure seldom aborts due to failure of the numerical algorithm. The salient disadvantage of explicit integration, the price you pay for the simplicity of the method and its avoidance of the solution of equations, is the conditional stability of explicit methods. If the time step exceeds a critical value $\Delta t_{\text {crit }}$, the solution may grow unboundedly and will in any case be erroneous.

The critical time step is also called the stable time step. The critical time step for a model depends on the mesh and the material properties. For low order elements, we will show in Section $X$ that the critical time step for linear response is given by

$$
\begin{equation*}
\Delta t_{c r i t}=\min \frac{\ell_{e}}{c_{e}} \tag{6.2.10}
\end{equation*}
$$

where $\ell_{e}$ is a characteristic length of element $e$ and $c_{e}$ the wavespeed of element $e$. Thus the critical time step decreases with mesh refinement and increasing stiffness of the material. The cost of an explicit simulation is independent of the frequency content which is of interest and depends only on the size of the model and the time of the simulation relative to the critical time step given by (6.2.10).

The time step is calculated in the flowchart on an element basis. For each element, a critical time step is calculated, and if it is smaller than that calculated for all previous elements in that time step, it is reset. The theoretical justification for setting the critical time step on an element basis and other approaches are described in Section 6.??.

### 6.3 EQUILIBRIUM SOLUTIONS AND IMPLICIT TIME INTEGRATION.

6.3.1. Equilibrium and Transient Problems. We will combine the description of the solution of the equilibrium equations with time integration by implicit methods because they share many common features. To begin, we write
the discrete momentum equation at time step $n+1$ in a form applicable to both equilibrium and dynamic problems:
$0=\mathbf{r}\left(\mathbf{d}^{n+1}, t^{n+1}\right)=s_{D} \mathbf{M} \ddot{\mathbf{d}}^{n+1}-\mathbf{f}^{n+1}=s_{D} \mathbf{M} \mathbf{a}^{n+1}-\mathbf{f}^{e x t}\left(\mathbf{d}^{n+1}, t^{n+1}\right)+\mathbf{f}^{i n t}\left(\mathbf{d}^{n+1}\right)$
where $s_{D}$ is a switch which is set by:

$$
s_{D}= \begin{cases}0 & \text { for a static(equilibrium) problem }  \tag{6.3.2}\\ 1 & \text { for a dynamic(transient) problem }\end{cases}
$$

The column matrix $\mathbf{r}\left(\mathbf{d}^{n+1}, t^{n+1}\right)$ is called a residual. When $s_{D}=0$, the above are the equilibrium equations at the next step. In addition, the displacement boundary conditions must be met; these can be written as a set of $n_{c}$ nonlinear algebaric equations

$$
\begin{equation*}
\mathcal{G}_{i}\left(\mathbf{d}^{n+1}\right)=g_{i}, \quad i=1 \text { to } n_{c} \tag{6.3.2b}
\end{equation*}
$$

Differential and integral constraints are put in discrete form by using discretizations of the derivatives and integrals, respectively. In most cases the displacement boundary conditions are linear algebraic equations, but we have written the general form (6.3.2b) because complex boundary conditions are often needed in nonlinear problems.

When the accelerations vanish or are negligible, a system is in equilibrium and the solution of the resulting equations is called an equilibrium solution. The equilibrium equations are given by (6.3.1) with $s_{D}=0$ :

$$
\begin{equation*}
0=\mathbf{r}\left(\mathbf{d}^{n+1}, t^{n+1}\right)=\mathbf{f}^{i n t}\left(\mathbf{d}^{n+1}, t^{n+1}\right)-\mathbf{f}^{e x t}\left(\mathbf{d}^{n+1}, t^{n+1}\right) \tag{6.3.3}
\end{equation*}
$$

In equilibrium problems, the residuals correspond to the out-of-balance forces; problems in which the accelerations can be neglected are called static problems.

The governing equations for both the implicit update of the equations of motion and the equilibrium equations are a set of nonlinear algebraic equations in the nodal displacements, $\mathbf{d}^{n+1}$. In equilibrium problems with rate-independent materials, $t$ need not be the real time. Instead it can be any monotonically increasing parameter which describes the changing load. If the constitutive equation is a differential or integral equation, it must also be discretized in time to obtain a set of algebraic equations for the system.
6.3.2a. Newmark $\beta$-equations. We will now show that the discrete equations obtained with an implicit time integrator are nonlinear algebraic equations in the unknowns $\mathbf{d}^{n+1}$. For this purpose we consider a popular class of time integrators called the Newmark $\beta$-method. In this time integration formula, the updated displacements and velocities are given by

$$
\begin{equation*}
\mathbf{d}^{n+1}=\tilde{\mathbf{d}}^{n}+\beta \Delta t^{2} \mathbf{a}^{n+1} \tag{6.3.4}
\end{equation*}
$$

$$
\begin{align*}
& \tilde{\mathbf{d}}^{n}=\mathbf{d}^{n}+\Delta t \mathbf{v}^{n}+\frac{\Delta t^{2}}{2}(1-2 \beta) \mathbf{a}^{n}  \tag{6.3.5}\\
& \mathbf{v}^{n+1}=\tilde{\mathbf{v}}^{n}+\gamma \Delta t \mathbf{a}^{n+1}  \tag{6.3.6}\\
& \tilde{\mathbf{v}}^{n}=\mathbf{v}^{n}+(1-\gamma) \Delta t \mathbf{a}^{n} \tag{6.3.7}
\end{align*}
$$

Here $\beta$ and $\gamma$ are parameters whose useful values are summarized in Box 6.2. In writing the time integration formulas, we have segregated the historical values of the nodal variables, i.e. those pertaining to time step $n$, in $\tilde{\mathbf{v}}^{n}$ and $\tilde{\mathbf{d}}^{n}$. The resulting formulas correspond to the predictor-corrector form given by Hughes and Liu( ). This segregation of the historical terms is convenient for the algebraic operations which follow and for the construction of explicit-implicit time integration procedures.

Equation (6.3.4) can be solved for the updated accelerations for $\beta>0$, giving

$$
\begin{equation*}
\mathbf{a}^{n+1}=\frac{1}{\beta \Delta t^{2}}\left(\mathbf{d}^{n+1}-\tilde{\mathbf{d}}^{n+1}\right) \tag{6.3.8}
\end{equation*}
$$

Substituting (6.3.8 ) into (6.3.1) gives

$$
\begin{equation*}
0=\mathbf{r}=\frac{s_{D}}{\beta \Delta t^{2}} \mathbf{M}\left(\mathbf{d}^{n+1}-\tilde{\mathbf{d}}^{n}\right)-\mathbf{f}^{e x t}\left(\mathbf{d}^{n+1}, t^{n+1}\right)+\mathbf{f}^{i n t}\left(\mathbf{d}^{n+1}, t^{n+1}\right) \tag{6.3.9}
\end{equation*}
$$

which is a set of nonlinear algebraic equations in the nodal displacements $\mathbf{d}^{n+1}$. Eq.(6.3.9) applies to both the static and dynamic problems. Therefore, in both cases we consider the discrete problem to be

$$
\begin{equation*}
\text { find } \mathbf{d}^{n+1} \text { so that } \mathbf{r}\left(\mathbf{d}^{n+1}\right)=\mathbf{0} \text { subject to } \mathbf{g}\left(\mathbf{d}^{n+1}\right)=0 \tag{6.3.10}
\end{equation*}
$$

where $\mathbf{r}\left(\mathbf{d}^{n+1}\right)$ is given by Eq. (6.3.9).
6.3.3. Newton's Method. The most widely used and most robust method for the solution of the nonlinear algebraic equations (6.3.9) is Newton's method. The method is often called the Newton-Raphson method in computational mechanics. It is identical to the Newton method taught in introductory calculus courses.

We first illustrate the Newton method for one equation in one unknown $d$ without a displacement boundary condition. It is then generalized to an arbitrary number of unknowns. For the case of one unknown, (6.3.9) reduces to a single nonlinear algebraic equation

$$
\begin{equation*}
r\left(d^{n+1}, t^{n+1}\right)=\frac{s_{D}}{\beta \Delta t^{2}} M\left(d^{n+1}-\dot{d}^{n}\right)-f\left(d^{n+1}, t^{n+1}\right)=0 \tag{6.3.11}
\end{equation*}
$$

The solution of (6.3.11) by Newton's method is an iterative procedure. The iteration number is indicated by Greek subsript: $d_{v}^{n+1}$ is the $v$ th iteration at time step $n+1$; when there is no chance for confusion, the time step number will be omitted.

To begin the iterative procedure, a starting value for the unknown must be chosen; usually the value of the solution $d^{n}$ from the last time step is used, so $d_{0}^{n+1} \equiv d^{n} \quad$ Taking a Taylor expansion of the residual about the current value of the nodal displacement, $d_{v}$ and setting the resulting residual equal to zero:

$$
\begin{equation*}
0=r\left(d_{v+1}, t^{n+1}\right)=r\left(d_{v}, t^{n+1}\right)+\frac{\partial r\left(d_{v}, t^{n+1}\right)}{\partial d} \Delta d+O\left(\Delta d^{2}\right) \tag{6.3.12}
\end{equation*}
$$

where

$$
\begin{align*}
& \Delta d=d_{v+1}-d_{v}  \tag{6.3.12b}\\
& r\left(d_{v}, t^{n+1}\right)=M a\left(d_{v}\right)+f^{i n t}\left(d_{v}, t^{n+1}\right)-f^{e x t}\left(d_{v}, t^{n+1}\right) \tag{6.3.13}
\end{align*}
$$

If the terms which are higher order in $\Delta d$ than linear are dropped, then (6.3.12) gives a linear equation for $\Delta d$ :

$$
\begin{equation*}
0=r\left(d_{v}, t^{n+1}\right)+\frac{\partial r\left(d_{v}, t^{n+1}\right)}{\partial d} \Delta d \tag{6.3.14}
\end{equation*}
$$

Note that in the Taylor expansion, the residual is written in terms of the time $t^{n+1}$. The time-dependence of the residual at constant nodal displacements is usually known. For example, if the tractions and body forces are given as functions of time, then the time dependent part of the nodal forces is known at time $t^{n+1}$ at the beginning of the iterations. Therefore the residual is always computed at time $t^{n+1}$. The above is called a linear model of the nonlinear equations. The linear model is the tangent to the nonlinear residual function; the process of obtaining the linear model is called linearization.

Equation (6.3.14) is often called a linear model of the nonlinear equations, Schnabel (?). Solving this linear model for the incremental displacements gives

$$
\begin{equation*}
\Delta d=-\left(\frac{\partial r\left(d_{v}\right)}{\partial d}\right)^{-1} r\left(d_{v}\right) \tag{6.3.15}
\end{equation*}
$$

In the Newton procedure, the solution to the nonlinear equation is obtained by iteratively solving a sequence of linear models (6.3.15). The new value for the unknown in each step of the iteration is obtained by rewriting Eq. (6.3.12b) as

$$
\begin{equation*}
d_{v+1}=d_{v}+\Delta d \tag{6.3.16}
\end{equation*}
$$

The procedure is illustrated in Fig. 6.1. The process is continued until the solution is obtained with the desired level of accuracy.


Fig. 6.1. Linear models for a nonlinear equation $r(d)=0$.
6.3.4. Newton's Method for $\boldsymbol{n}$ Unknowns. The generalization of this procedure to $n_{D O F}$ unknowns is accomplished by replacing the above scalar equations by matrix equations. The counterpart of Eq. (6.3.12) becomes

$$
\begin{align*}
& \mathbf{r}\left(\mathbf{d}_{v}\right)+\frac{\partial \mathbf{r}\left(\mathbf{d}_{v}\right)}{\partial \mathbf{d}} \Delta \mathbf{d}+O\left(\Delta \mathbf{d}^{2}\right)=0 \\
& \text { or } \\
& r_{a}\left(\mathbf{d}_{v}\right)+\sum_{b=1}^{n_{D O F}} \frac{\partial r_{a}\left(\mathbf{d}_{v}\right)}{\partial d_{b}} \Delta d_{b}+O\left(\Delta d_{b}\right)^{2}=0 \tag{6.3.17}
\end{align*}
$$

The matrix $\partial \mathbf{r} / \partial \mathbf{d}$ is called the Jacobian matrix and will be denoted by $\mathbf{A}$ :

$$
\begin{equation*}
\mathbf{A}=\frac{\partial \mathbf{r}}{\partial \mathbf{d}}, \quad \text { or } \quad A_{a b}=\frac{\partial r_{a}}{\partial d_{b}} \tag{6.3.18}
\end{equation*}
$$

Using (6.3.17) and dropping terms in higher order than linear, Eqs. (6.3.16) can be rewritten as

$$
\begin{equation*}
\mathbf{r}+\mathbf{A} \Delta \mathbf{d}=\mathbf{0} \tag{6.3.19}
\end{equation*}
$$

which is the linear model of the nonlinear equations. The linear model is difficult to picture for problems with more than one unknown, since $\mathbf{r}(\mathbf{d})$ maps $\mathfrak{R}^{n}$ to $\Re^{n}$, Figure 6.2 shows the first component of the residual for a function of two unknowns. The linear model is a plane tangent to the nonlinear function $r_{1}\left(d_{1}, d_{2}\right)$. The other residual component is another nonlinear function $r_{2}\left(d_{1}, d_{2}\right)$, which is not drawn.


Figure 6.2. Depiction of a residual component $r_{1}$ as a function of $d_{1}$ and $d_{2}$ and the tangent plane.
The increment in the nodal displacements in the Newton iterative procedure is obtained by solving (6.3.18), which gives

$$
\begin{equation*}
\Delta \mathbf{d}=-\mathbf{A}^{-1} \mathbf{r}\left(\mathbf{d}_{v}, t^{n+1}\right) \tag{6.3.20}
\end{equation*}
$$

The increment in the nodal displacements is obtained from this system of linear algebraic equations. The solution of these equations is discussed in Section X. Once the increments in nodal displacements have been obtained, the new values of the nodal displacements are obtained by

$$
\begin{equation*}
\mathbf{d}_{v+1}=\mathbf{d}_{v}+\Delta \mathbf{d} \tag{6.3.21}
\end{equation*}
$$

The new displacement is checked for convergence, see Section 6.3.7. If the convergence criterion is not met, a new linear model is constructed and used to find another increment in the nodal displacements. The procedure is repeated until the convergence criterion is satisfied.

In computational mechanics, the Jacobian is called the effective tangent stiffness matrix and the contributions of the inertial, internal and external nodal forces are linearized separately. From (6.3.9) we can write

$$
\begin{equation*}
\mathbf{A}=\frac{\partial \mathbf{r}}{\partial \mathbf{d}}=\frac{s_{D}}{\beta \Delta t^{2}} \mathbf{M}+\frac{\partial \mathbf{f}^{i n t}}{\partial \mathbf{d}}-\frac{\partial \mathbf{f}^{e x t}}{\partial \mathbf{d}} \tag{6.3.22}
\end{equation*}
$$

where we have used the fact that the mass matrix in a Lagrangian mesh is constant in time and (6.3.4). The Jacobian of the internal nodal forces is called the tangent stiffness matrix and will be denoted by $\mathbf{K}^{\text {int }}$ :

$$
\begin{equation*}
K_{a b}^{i n t}=\frac{\partial f_{a}^{i n t}}{\partial d_{b}} \quad K_{i j J J}^{i n t}=\frac{\partial f_{i I}^{i n t}}{\partial u_{j J}} \quad \mathbf{K}^{i n t}=\frac{\partial \mathbf{f}^{i n t}}{\partial \mathbf{d}} \tag{6.3.23}
\end{equation*}
$$

The tangent stiffness matrix is shown above in three forms. The Jacobian matrix of the external nodal forces is called the load stiffness matrix and denoted by

$$
\begin{equation*}
K_{a b}^{e x t}=\frac{\partial f_{a}^{e x t}}{\partial d_{b}} \quad K_{i l j J}^{e x t}=\frac{\partial f_{i I}^{e x t}}{\partial u_{j J}} \quad \mathbf{K}^{e x t}=\frac{\partial \mathbf{f}^{e x t}}{\partial \mathbf{d}} \tag{6.3.24}
\end{equation*}
$$

The development of these matrices is the topic of linearization and is treated in Sections 6.4 and 6.5. Using these definitions, the Jacobian matrix (6.3.22) can be written as

$$
\begin{equation*}
\mathbf{A}=\frac{s_{D}}{\beta \Delta t^{2}} \mathbf{M}+\mathbf{K}^{i n t}-\mathbf{K}^{e x t} \tag{6.3.25}
\end{equation*}
$$

This Jacobian matrix applies to both dynamic and equilibrium problems with the dynamic switch $s_{D}$ set by (6.3.2).

The Jacobians in (6.3.23-24) can be used to relate differentials of the nodal forces to differentials of the nodal displacements by

$$
\begin{equation*}
d \mathbf{f}^{i n t}=\mathbf{K}^{i n t} d \mathbf{d} \quad d \mathbf{f}^{e x t}=\mathbf{K}^{\text {ext }} d \mathbf{d} \quad d \mathbf{r}=\mathbf{A} d \mathbf{d} \tag{6.3.26}
\end{equation*}
$$

The matrices which relate finite increments of nodal displacements to increments of nodal forces differ from the above. We will use a

$$
\begin{equation*}
\Delta \mathbf{f}^{i n t}=\mathbf{K}_{\Delta}^{i n t} \Delta \mathbf{d} \quad \Delta \mathbf{f}^{e x t}=\mathbf{K}_{\Delta}^{e x t} \Delta \mathbf{d} \quad \Delta \mathbf{r}=\mathbf{A}_{\Delta} \Delta \mathbf{d} \tag{6.3.27}
\end{equation*}
$$

The matrix $\mathbf{K}_{\Delta}^{\text {int }}$ is called a secant stiffness and $\mathbf{A}_{\Delta}$ the secant Jacobian. The secant stiffnes and secant Jacobian depend on the magnitude and direction of $\Delta \mathbf{d}$. This can easily be seen in one dimension as illustrated in Fig. 6.3, which shows secants for various stepsizes and two directions (there are only two in a function of a single variable). The tangent and secant Jacobians are identical only in the limit as $\Delta \mathbf{d} \rightarrow 0$; for finite increments, the secant stiffness in (6.3.27) differs from the tangent stiffness in (6.3.23).


Figure 6.3. Secant Jacobians for various step sizes and directions.
Conservative Problems (Stationary Points). It is useful at this point to examine the discrete problem corresponding to the stationary principle described in Section 4.9.3. This stationary principle only applies to conservative equilibrium problems, but it is nevertheless provides insight into the character of nonlinear problems. An equilibrium solution is a stationary point of the potential, so by enforcing the conditions that the derivative of the potential vanish and using (4.9.29-30) and the definition of the residual (6.3.3) we have

$$
\begin{equation*}
0=\mathbf{r}=-\frac{\partial W}{\partial \mathbf{d}}=\frac{\partial W^{i n t}}{\partial \mathbf{d}}-\frac{\partial W^{e x t}}{\partial \mathbf{d}}=\mathbf{f}^{i n t}-\mathbf{f}^{e x t} \tag{6.3.28}
\end{equation*}
$$

A solution is a stable equilibrium solution if it corresponds to a minimum of the potential energy. Thus stable equilibrium solutions can be found by minimizing the potential $W$. The situation is depicted in Fig. 6.3, which shows the local behavior of a potential of two generalized displacements and the contours for this potential. The residual is the negative of the gradient of the potential (note the sign in the above.)

The linear model for (6.3.28) is (see 6.3.17-18)

$$
\begin{align*}
& -\mathbf{r}_{v}=\frac{\partial \mathbf{r}}{\partial \mathbf{d}} \Delta \mathbf{d}=-\frac{\partial^{2} W}{\partial \mathbf{d} \partial \mathbf{d}} \Delta \mathbf{d}=\mathbf{A} \Delta \mathbf{d} \text { where } A_{a b}=\frac{\partial^{2} W}{\partial d_{a} \partial d_{b}} \text { or } \mathbf{A}=\frac{\partial^{2} W}{\partial \mathbf{d} \partial \mathbf{d}} \\
& -r_{a_{v}}=\frac{\partial r_{a}}{\partial d_{b}} \Delta d_{b}=-\frac{\partial^{2} W}{\partial d_{a} \partial d_{b}} \Delta d_{b}=A_{a b} \Delta d_{b} \text { where } A_{a b}=\frac{\partial^{2} W}{\partial d_{a} \partial d_{b}} \tag{6.3.29}
\end{align*}
$$

The matrix $\mathbf{A}$ when it arises from the second derivatives of a potential is called a Hessian matrix. It is identical to the Jacobian, so

$$
\begin{equation*}
\mathbf{A}=\mathbf{K}^{i n t}-\mathbf{K}^{\text {ext }} \tag{6.3.30}
\end{equation*}
$$

The linearized equations for a conservative system are

$$
\left(\mathbf{K}^{i n t}-\mathbf{K}^{e x t}\right) \Delta \mathbf{d}=-\mathbf{r}
$$

The above are identical to Eq. (6.3.19) except that the mass matrix is omitted, since dynamic effects cannot be included in a conservative problem. However, when the problem is posed as a minimization problem, many techniques not directly applicable to linear models, such as the method of steepest descent, can be applied to the problem. Thus, viewing the solution of the residual equations as a minimization problem is helpful in many cases.

### 6.3.2b. $\alpha$-METHOD EQUATIONS

The $\alpha$-method, also known as Hilber-Hughes-Taylor (HHT) method [??], was introduced to improve numerical dissipation for high frequencies in the Newmark- $\beta$ method. The Newmark- $\beta$ formulas, Eqs. (6.3.4) - (6.3.7), remain the same, wheras the time-discrete equation of motion is modified as follows (cf Eq. 6.3.1)

$$
\begin{equation*}
0=\mathbf{r}\left(\mathbf{d}^{n+1}, t^{n+1}\right)=s_{D} \mathbf{M} \mathbf{a}^{n+1}-\mathbf{f}^{e x t}\left(\mathbf{d}^{n+\alpha}, t^{n+1}\right)+\mathbf{f}^{i n t}\left(\mathbf{d}^{n+\alpha}\right) \tag{6.6.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{d}^{n+\alpha}=(1+\alpha) \mathbf{d}^{n+1} \alpha \mathbf{d}^{n} \tag{6.6.2}
\end{equation*}
$$

It is noted that in the case of a linear analysis, the internal force vector becomes:

$$
\begin{equation*}
\mathbf{f}^{i n t}\left(\mathbf{d}^{n+\alpha}\right)=\mathbf{K} \mathbf{d}^{n+\alpha}=(1+\alpha) \mathbf{K} \mathbf{d}^{n+1}-\alpha \mathbf{K} \mathbf{d}^{n} \tag{6.6.3}
\end{equation*}
$$

which is exactly the HHT method presented in [??]. Follow the $\alpha$-method stability analysis, unconditional stability is achieved by setting the following parameters:

$$
\begin{equation*}
\alpha \in\left[-\frac{1}{3}, 0\right] ; \gamma=\frac{(1-2 \alpha)}{2} ; \text { and } \beta=\frac{(1-\alpha)^{2}}{4} \tag{6.6.4}
\end{equation*}
$$

If $\alpha=0$, the trapezoidal rule is obtained.
Remark: Although there is no stability analysis in the literature for a nonlinear setting (i.e., with Eq. (6.6.1)), a linearized stability analysis will yield the same stability result as in Eq. (6.6.4).

Following the same procedure given in Section 6.3.2, the discrete problem as stated in Eq. (6.3.10) is revised as:
find $\mathbf{d}^{n+1}$ so that $\mathbf{r}\left(\mathbf{d}^{n+1}\right)=0$ as shown in Eqs. (6.6.1) - (6.6.2), subject to $\mathbf{g}\left(\mathbf{d}^{n+1}\right)=0$.

In order to define the Jacobian matrices and incremental nodal displacements given in Eqs. (6.3.18) through (6.3.25), the following linearized displacement equations are defined (cf. Eq. (6.3.21)):

$$
\begin{equation*}
\mathbf{d}_{v+1}^{n+\alpha} \stackrel{\text { def }}{=}(1+\alpha)\left(\mathbf{d}_{v}^{n+1}+\Delta \mathbf{d}\right)-\alpha \mathbf{d}^{n}=\mathbf{d}_{v}^{n+\alpha}+\Delta \tilde{\mathbf{d}} \tag{6.6.5a}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{d}_{v+1}^{n+\alpha}=(1+\alpha) \mathbf{d}_{v}^{n+1}-\alpha \mathbf{d}^{n} \tag{6.6.5b}
\end{equation*}
$$

and

$$
\text { for } v=0
$$

$$
\begin{equation*}
\Delta \tilde{\mathbf{d}}=(1+\alpha) \Delta \mathbf{d} \quad \mathbf{a}_{0}^{n+1}=\mathbf{0}, \mathbf{d}_{0}^{n+1}=d^{n}+\Delta t v^{n}+\frac{\Delta t^{2}}{2}(1-2 \beta) \mathbf{a} \tag{6.6.5c}
\end{equation*}
$$

With the above definitions, the linearized Jacobian matrix equations becomes: (cf. Eq. (6.3.17))

$$
\begin{equation*}
\mathbf{r}\left(\mathbf{d}_{v}^{n+\alpha}\right)+\frac{\partial \mathbf{r}\left(\mathbf{d}_{v}^{n+\alpha}\right)}{\partial \mathbf{d}} \Delta \mathbf{d}+0\left(\Delta \mathbf{d}^{2}\right)=0 \tag{6.6.6}
\end{equation*}
$$

The Jacobian matrix or the effective tangent stiffness matrix (cf. Eq. (6.3.22)) can be shown to be

$$
\begin{equation*}
\mathbf{A}=\frac{\partial \mathbf{r}\left(\mathbf{d}_{v}^{n+\alpha}\right)}{\partial \mathbf{d}}=\frac{s_{D}}{\beta \Delta t^{2}} \mathbf{M}+(1+\alpha) \frac{\partial \mathbf{f}^{i n t}\left(\mathbf{d}_{v}^{n+\alpha}\right)}{\partial \mathbf{d}}-(1+\alpha) \frac{\partial \mathbf{f}^{e x t}\left(\mathbf{d}_{v}^{n+\alpha}\right)}{\partial \mathbf{d}} \tag{6.6.8}
\end{equation*}
$$

The rest of the formulation remains the same.
6.3.5. Implementation of Newton Method. Flowcharts for implicit integration and equilibrium solutions are given in Boxes 6.3 and 6.4. Both the dynamic problem and the equilibrium problem are solved by time-stepping: the external loads and other conditions are described as functions of time, which is incremented over the range of interest. In equilibrium problems, the time is often
replaced by a monotonically increasing parameter. Solutions of equilibrium processes obtained in this manner are called incremental solutions.

The flowchart shows a procedure often called a full Newton algorithm, where the Jacobian matrix is inverted in every iteration of the procedure. Many programs use a modified Newton algorithm, in which the Jacobian is only triangulated at the beginning of the iterations or intermittently during the iteration. For, example, in a modified Newton procedure the Jacobian may be triangulated only when the ietrative procedure does not seem to be converging well. These modified schemes are faster but less robust.

The flowcharts begin with the imposition of the intial conditions. The initial conditions can be handled exactly as in explicit methods. The initial displacements are considered to be zero. The initial accelerations are computed as shown in steps 2 and 3 .

The displacements $\mathbf{d}^{n+1}$ for each time step are obtained by the iterative Newton procedure. To begin the iterative procedure, a starting value of $\mathbf{d}$ is needed; usually the solution from the preceding step is used. The residual is then calculated for this starting value. In an equilibrium solution, the residual depends only on the internal and external nodal forces. and is obtained in the module getf. This module, getf, is the same as in the explicit procedure, Box 6.1, except that the calculation of the stable time step is omitted, so it is not repeated. In transient implicit solutions, the residuals also depend on the accelerations.

## Box 6.3

## Flowchart for Implicit Time Integration

1. Initial conditions \& initialization of parameters:

$$
\text { set } \mathbf{v}^{0}, \sigma^{0} ; \mathbf{d}^{0}=\mathbf{0}, n=0, t=0 ; \text { compute } \mathbf{M}
$$

2. get $\mathbf{f}^{0}=\mathbf{f}\left(\mathbf{d}^{0}, 0\right)$
3. compute initial accelerations $\mathbf{a}^{n}=\mathbf{M}^{-1} \mathbf{f}^{n}$
4. estimate next solution $\mathbf{d}$ : $\mathbf{d}=\mathbf{d}^{n}$
5. Newton iterations for time step $n+1$
a. get $f$ computes $\mathbf{f}\left(\mathbf{d}, t^{n+1}\right)$
b. $\mathbf{a}^{n+1}=\frac{1}{\beta \Delta t^{2}}\left(\mathbf{d}-\tilde{\mathbf{d}}^{n}\right), \quad \mathbf{v}^{n+1}=\tilde{\mathbf{v}}^{n}+\gamma \Delta t \mathbf{a}^{n+1}$, $\quad$ see Eqs. (6.3.4-6.3.7)
c. $\mathbf{r}=\mathbf{M a}^{n+1}-\mathbf{f}$
d. compute Jacobian $\mathbf{A ( d )}$
e. modify $\mathbf{A}(\mathbf{d})$ for essential boundary conditions
f. solve linear equations $\Delta \mathbf{d}=\mathbf{A}^{-1} \mathbf{r}$
g. $\mathbf{d} \leftarrow \mathbf{d}+\Delta \mathbf{d}$
h. check error criterion; if not met, go to step 5a
6. update displacements, counter and time: $\mathbf{d}^{n+1}=\mathbf{d}, n \leftarrow n+1, t \leftarrow t+\Delta t$
7. check energy balance
8. output, if simulation not complete, go to 3

## Box 6.4 <br> Flowchart for Equilibrium Solution

1. Initial conditions and initialization: set $\mathbf{u}^{0}=\mathbf{0} ; \sigma^{0} ; n=0$;
2. Newton iterations for load increment $n+1$
a. getf computes $\mathbf{f}\left(\mathbf{d}, t^{n+1}\right) ; \mathbf{r}=\mathbf{f}\left(\mathbf{d}, t^{n+1}\right)$
b. compute $\mathbf{A}(\mathbf{d})$
c. modify $\mathbf{A}(\mathbf{d})$ for essential boundary conditions
d. solve linear equations $\Delta \mathbf{d}=\mathbf{A}^{-1} \mathbf{r}$
e. $\mathbf{d} \leftarrow \mathbf{d}+\Delta \mathbf{d}$
f. check error criterion; if not met, go to 2 a
3. update displacements, counter and time: $\mathbf{d}^{n+1}=\mathbf{d}, n \leftarrow n+1, t \leftarrow t+\Delta t$
4. output, if simulation not complete, go to 2

The Jacobian matrix in this algorithm is then calculated based on the latest state of the body. In some algorithms, the Jacobian for the last converged solution is used for all the iterations or the Jacobian is only recomputed intermittenly during the iterations; these are known as modified Newton methods. Simple essential boundary conditions, such as homogeneous displacement conditions, can be enforced by modifying the Jacobian matrix. The equation corresponding to the vanishing displacement component is either omitted or replaced by a dummy equation that the component vanishes by zeroing the cooresponding row and column and putting a one on the diagonal of the Jacobian. For more complex algebraic constraints, Lagrange multipliers methods or penalty methods are used: these are described in Section 6.?.

### 6.3.6. Equilibrium Solutions Based on Stationary Potential

Energy. In Chapter 4 we saw that when the system is conservative, i.e. when the stresses and external loads are derivable from a potential, then the equilibrium problem can be posed as the determination of the stationary points of the energy. Such problems are called conservative. Stable equilibrium solutions correspond to local minima of the potential energy.

Consequently, stable solutions for conservative problems can be found by minimization techniques. The discrete problem is then: for any time $t$ (the time parametrizes the external load):

$$
\begin{equation*}
\min W(\mathbf{d}, t) \text { subject to } g_{I}(\mathbf{d})=0 \quad I=1 \text { to } n_{c} \tag{6.3.31}
\end{equation*}
$$

where $g_{I}(\mathbf{d})=0$ are $n_{c}$ discrete constraints on the system. These must be linear algebraic constraints. If they involve differentials or integrals, they must be converted to algebraic from by time discretization. Displacement boundary conditions are often imposed as auxiliary consraints of this type. Often the essential boundary conditions can be met by simply eliminating nodal displacements from the unknowns. If both linear stable and linear unstable
solutions are desired, then the the stationary points of $\mathcal{W}(\mathbf{d}, t)$ must be found. The discrete problem is then

$$
\text { find } \mathbf{d} \text { so that } \frac{\partial W(\mathbf{d})}{\partial \mathbf{d}}=-\mathbf{f}=\mathbf{r}=\mathbf{0} \quad \text { subject to } g_{I}(\mathbf{d})=0 \quad I=1 \text { to } n_{c} \text { (6.3.32) }
$$

Solutions to these equations for loads which vary as a function of the parameter $t$, which could be time but need not be, appear as branches (lines) in the space of the nodal displacement components. Some examples are given in Section 6.??.

In the above we have indicated that the derivatives of the potential with respect to the nodal displacements is the negative of the nodal forces, which are in turn equal to the residuals. Viewing an equilibrium solution as the determination of the stationary points of a potential provides substantial insight, particularly when the stability of a solution is of interest. This is pursued further in Section ??. As can be seen from a comparison of Eqs. (6.3.1) and (6.3.27), the equations for a stationary point are identical to the discrete equations derived previously. These methods are not applicable to dynamic problems.
6.3.8 Convergence Criteria. The termination of the iterative procedure in implicit and equilibrium solutions by the Newton method is determined by convergence criteria. These criteria pertain to the convergence of the discrete solution to the equations $\mathbf{r}\left(\mathbf{d}^{n}, t^{n}\right)=0$, not the convergence of the discrete solution to the solution of the partial differential equations. Three types of convergence criteria are used to control the iterations:

1. criteria based on the magnitude of the residual $\mathbf{r}$;
2. criteria based on the magnitude of the displacement increments $\Delta \mathbf{d}$;
3. energy error criteria.

Usually an $\ell_{2}$ norm of the vectors is used for the first two criteria. The criteria then are:
residual error criterion:

$$
\begin{equation*}
\mid \mathbf{r} \|_{\ell_{2}}=\left(\sum_{a=1}^{n_{D O F}} r_{a}^{2}\right)^{\frac{1}{2}} \leq \varepsilon \max \left(\left\|\left.\mathbf{f}^{e x t}\right|_{\ell_{2}}, \mid \mathbf{f}^{i n t}\right\|_{\ell_{2}},\|\mathbf{M} \mathbf{a}\|_{\ell_{2}}\right) \tag{6.3.28}
\end{equation*}
$$

displacement increment error criterion:

$$
\begin{equation*}
\left\lvert\, \Delta \mathbf{d}\left\|_{\ell_{2}}=\left(\sum_{a=1}^{n_{D O F}} \Delta d_{a}^{2}\right)^{\frac{1}{2}} \leq \varepsilon\right\| \mathbf{d}\right. \|_{\ell_{2}} \tag{6.3.29}
\end{equation*}
$$

The $\ell_{2}$ norm, which has been indicated in the above, is the probably most suitable when the mean error over all degrees of freedom is to be controlled, but a maximum norm can also be used. A maximum norm would limit the maximum error at any node. The terms on the right-hand side of Eqs. (6.3.28) and (6.3.29) are scaling factors. Without these, the criterion would depend on the parameters
of the problem. The error tolerancee determines the precision with which the displacements are calculated before terminating the iterative procedure; when $\varepsilon=10^{-3}$, the mean accuracy of the nodal displacements is in the third significant digit when the $\ell_{2}$ norm is used. The convergence tolerance determines the speed and accuracy of a calculation. If the criterion is too coarse, the solution may be quite inaccurate. On the other hand, a criterion which is too tight results in unnecessary computations.

The energy convergence criterion measures the energy flow to the system resulting from the residual, which is like an error in energy. It is given by

$$
\begin{equation*}
\Delta \mathbf{d}^{T} \mathbf{r}=\Delta d_{a} r_{a} \leq \varepsilon \max \left(W^{\text {ext }}, W^{\text {int }}, W^{k i n}\right) \tag{6.3.30}
\end{equation*}
$$

where the computation of the energies used for scaling the criterion is described in Section 6.?. The left hand side in the above represents an error in the energy, since a nonzero residual is an error in the forces on the system.
6.3.7. Convergence and Robustness of Newton Iteration. The rate of the convergence of the iterations in the Newton method is quadratic when the Jacobian matrix A satisfies certain conditions. These conditions may roughly be described as follows:

1. the Jacobian $\mathbf{A}$ should be a sufficiently smooth function of $\mathbf{d}$;
2. the Jacobian $\mathbf{A}$ should be regular (invertable) and well-conditioned in the entire domain in the displacement space that the iterative procedure traverses.

Quadratic convergence means that the $\ell_{2}$ norm of the difference between the solution and the iterate $\mathbf{d}_{v}$ decreases quadratically in each iteration:

$$
\begin{equation*}
\left|\mathbf{d}_{v+1}-\mathbf{d}\right| \leq c\left\|\mathbf{d}_{v}-\mathbf{d}\right\|^{2} \tag{6.3.31}
\end{equation*}
$$

where $c$ is a constant that depends on the nonlinearity of the problem and $\mathbf{d}$ is the solution to the nonlinear algebraic equations. Thus the convergence of the Newton algorithm is quite rapid when $\mathbf{A}$ meets the above conditions. The above gives the requirements for convergence only in broad terms and convergence has been proven for various conditions on $\mathbf{A}$. One set of conditions for quadratic convergence are: the residual must be continuously differentiable and the inverse of the Jacobian matrix must exist and be uniformly bounded in the neighborhood of the solution, Dennis and Schnabel (1983, p 90).

These conditions are usually not satisfied by nonlinear finite element problems. For example, in an elastic-plastic material, the residual is not continuously differentiable when a discrete point changes from elastic to plastic or vice versa; therefore, the Jacobian is discontinuous. In a two degree of freedom problem, the discontinuities in the Jacobian appear as kinks in the contour plots for the residual components. This is illustrated in Example X. In the solution of contact-impact problems with Lagrange multiplier methods, the residual often lacks smoothness, as illustrated by Chapter 10. Thus the conditions for quadratic convergence of the Newton method are often not satisfied in engineering
problems. Yet, Newton's method is remarkably effective in engineering problems, although the rate of convergence often deteriorates. At this time, more robust methods are not available. In many problems, the conditions for quadratic convergence are satisfied; for example, the above conditions are satisfied in the response of a model with a Mooney-Rivlin material when the load is small enough so that the equilibirium solutions are stable.

Newton's method fails particularly often when applied to equilibrium problems. Since Eq. (6.3.3) are nonlinear algebraic equations, they can have multiple solutions and solutions in which are unstable. When the equilibrium path is unstable, the inverse of the Jacobian matrix is no longer regular at all points and the proof of quadratic convergence does not apply. The convergence of the Newton method often fails in the vicinity of unstable states. These types of problems are considered in the next Section.

In summary, Newton's method sometimes lacks robustness when applied to engineering problems. The robustness decreases as we increase the time step and appears more often in equilibrium solutions, since in the latter we lose the effect of the mass matrix. The mass matrix improves the conditioning of the Jacobian matrix because it is always positive definite, see Exercise X. As the time step increases, the beneficial effects of the mass matrix decrease since the coefficient of the mass matrix is inversely proportional to the square of the time step, as can be seen from Eq. (6.3.9). For many problems, a straightforward application of the Newton method will sometimes fail completely, and enhancements of the Newton method such as the arc length method, line search, and augmented Lagrangian, which are described in Section ?, are needed to solve the nonlinear algebraic equations.
6.3.8. Line Search. An effective way to increase the robustness of Newton methods when convergence is slow is to use the line search technique. The rationale behind line search is that the direction $\Delta \mathbf{d}$ found by the Newton method is often a good direction, but the step size is not optimal. It is cheaper to find the best point along this direction by several computations of the residual than to get a new direction by using a new Jacobian. Therefore, before proceeding to the next direction, the residual is minimized along the line $\mathbf{d}_{\text {old }}+\xi \Delta \mathbf{d}$ where $\mathbf{d}_{\text {old }}$ is the last iterate and $\xi>0$ is a parameter. In other words, we find the parameter $\xi$ so that $\mathbf{d}_{\text {old }}+\xi \Delta \mathbf{d}$ minimizes some measure of the residual. We can use as a measure of the measure of the residual its $\ell_{2}$ norm, as defined in Eq. (6.3.28), the maximum norm, i.e. the maximum absolute value of any component of the residual, or some other measure. Line search then involves the calculation of two or more residuals along the line and an interpolation of a measure of the residual. For example, if the $\ell_{2}$ norm is used, then

A measure for the residual which is frequently used in line search is based on the existence of a potential for the problem, i.e. on the solution by the stationary energy principle, Sections 4.9.3 and 6.3.6. For a conservative problem, the minimizer of the potential $W(\mathbf{d})$, along the line $\Delta \mathbf{d}$ is the point where the gradient of the function is orthogonal to the line. The residual is given in terms of a potential by

$$
\begin{equation*}
\frac{\partial W}{\partial \mathbf{d}}=\frac{\partial W^{i n t}}{\partial \mathbf{d}}-\frac{\partial W^{e x t}}{\partial \mathbf{d}}=\mathbf{f}^{i n t}-\mathbf{f}^{e x t}=\mathbf{r} \tag{6.3.32}
\end{equation*}
$$

where the above follows from Eqs. (4.9.34) and (6.3.3). When the residual is orthogonal to the incremental displacement

$$
\begin{equation*}
\Delta \mathbf{d}^{T} \mathbf{r}=0 \Rightarrow \Delta \mathbf{d}^{T} \frac{\partial W}{\partial \mathbf{d}}=0 \tag{6.3.33}
\end{equation*}
$$

the potential must be minimum (or be stationary) at that point. This is illustrated in Fig. 6.1, which shows the contours of the potential energy for a two degree-offreedom system and the residual of the nodal forces for several points along the line $\mathbf{d}_{\text {old }}+\xi \Delta \mathbf{d}$. As can be seen, the potential is minimum when the residual, i.e. the gradient of the potential, is normal to the line. The line search can then be conducted by minimizing $\Delta \mathbf{d}^{T} \mathbf{r}$.

This criterion can also be used for systems that are not conservative, since $\Delta \mathbf{d}^{T} \mathbf{r}$ does not involve the potential. Note that this measure of the residual is equivalent to the criterion for error in energy, Eq. (6.3.30).

Equation (6.3.33a) can also be derived directly by using the chain rule to expand the potential energy in the parameter $\xi$. This gives

$$
\begin{equation*}
\frac{d W(\xi)}{d \xi}=\frac{\partial W}{\partial \mathbf{d}} \cdot \frac{d \mathbf{d}}{d \xi}=0 \Rightarrow \mathbf{r}^{T} \Delta \mathbf{d}=0 \tag{6.3.34}
\end{equation*}
$$

where we have set the derivative of the potential energy with respect to the parameter $\xi$ equal to zero, since we are looking for the minimum of the potential along the line $\Delta \mathbf{d}$ parametrized by $\xi$. The second equation follows from (6.3.32) and

$$
\begin{equation*}
\frac{d \mathbf{d}}{d \xi}=\frac{d\left(\mathbf{d}_{o l d}+\xi \Delta \mathbf{d}\right)}{d \xi}=\Delta \mathbf{d} \tag{6.3.35}
\end{equation*}
$$

Once a measure of the residual has been chosen, the line search can be made with any of the methods for minimizing a function of a single parameter. The method of bisection or searches based on interpolation or combinations thereof can be used. Once the residual has been evaluated at two points, a quadratic fit can be made to the residual measure, since its value at $\xi=0$ is known to vanish. This quadratic fit can then be used to estimate the position of the minimum. The iteration along the line is terminated when the measure has been minimized to a suitable precision. Note that when the orthogonality condition (6.3.29) is used, it should be normalized like the error energy criterion is in Eq. (6.3.26).

### 6.3.9. Secant Methods to be inserted

6.3.10. Stability of Implicit methods. The advantage of an implicit method over an explicit method is that for linear transient, problems, suitable implicit integrators are unconditionally stable. The unconditional stability of implicit integrators has not been proven for all nonlinear systems, although results which deal with specific situations indicate that unconditional stability holds at least for certain nonlinear systems. In any case, experience indicates that the time steps which can be used with implicit integrators are much larger than those for explicit integration in many problems.

The major restrictions on the size of time steps in implicit methods arise from accuracy requirements and the decreasing robustness of the Newton procedure as the time step increases. The latter is particularly pronounced in problems with very rough response, such as contact-impact. With a large time step, the starting iterate may be far from the solution, so the possibility of failure of the Newton method to converge increases. Therefore small time steps are often used to improve the robustness of the Newton algorithm.

In return for their enhanced stability, implicit methods exact a significant price: implicit methods require the solution of nonlinear algebraic equations in each time step. The construction of the linearized algebraic equations for the Newton procedure is often quite involved. Furthermore, the storage of these equations requires significant amounts of memory. The memory requirements can be reduced substantially by iterative linear equation solvers (an iterative method within an iterative Newton method). In recent research, iterative solvers have been improved dramatically, so implicit solutions are feasible in many problems where they were prohibitive before, see Section ?. The robustness and speed of Newton methods has increased markedly over the past two decades, and we are certain that further improvements are imminent. Nevertheless, high cost and lack of robustness are still plague many implicit and equilibrium solutions.

### 6.4 LINEARIZATION

There are several different ways to linearize the discrete equations. In discussing the various linearization procedures, it is useful to keep in mind that the order in which linearization and spatial discretization are carried out does not matter (in mathematical terminology, the operations of linearization and spatial discretization are said to commute). This means that linearization of the semidiscrete equations of motion (6.2.x) gives rise to the same finite element equations as does the semi-discretization of a linearized weak form (we have not yet developed such forms, but they appear frequently in the literature). The choice between these two approaches is a matter of style. For completeness, we will consider both approaches.

In the linearization procedure, there are several possibilities:

1. Linearization is carried out before the stress-update algorithm (integration algorithm for the constitutive equation) is introduced; this gives rise to the so-called continuum tangent moduli which will be discussed below.
2. Linearization is carried out after the stress-update algorithm is introduced; this gives rise to the so-called algorithmic moduli.

These two distinct approaches yield different tangent stiffness matrices. The choice of which approach to use rests on practical considerations related to ease of implementation and on convergence of the iterative scheme. The first
approach, based on the continuum tangent modulus, is straightforward to implement. However, it can run into convergence difficulties, especially for elastic-plastic materials where the slope discontinuity at the yield point on the stress-strain curve requires small steps to assure convergence and to preserve accuracy.

The second approach, based on the algorithmic moduli, exhibits better convergence because, through linearization of the stress-update algorithm, it accounts for the change in slope associated with a finite increment of strain. One drawback of the method is that it is not always possible to derive explicit forms for the algorithmic moduli for complex constitutive relations. Numerical differentiation schemes are sometimes used to obtain the algorithmic moduli, and they introduce additional inaccuracies.

We first consider linearization of the discrete equations based on the continuum tangent moduli, which relate a stress rate to a strain rate. The resulting material tangent stiffness matrix is called the continuum tangent stiffness matrix.

A somewhat more mathematical approach to linearization based on directional derivatives is then presented and it is shown how the resulting expressions are equivalent to those obtained by using the procedure based on the material time derivative. This linearization procedure based on the directional derivative is then used to develop the linearized equations for the second approach discussed above, i.e, linearization of the weak form after introduction of the stress-update algorithm. Because the stress-update algorithm is introduced prior to linearization, the expression for the stress increment that appears in the linearization of the weak form is based on the linearized constitutive integration scheme and not on the continuum rate form of the constitutive relation. As a result, the material tangent stiffness differs from the continuum tangent stiffness and is referred to as the algorithmic modulus (sometimes referred to as the consistent tangent modulus because of the consistent linearization of the weak form and the stress-update algorithm). Examples of the algorithmic modulus for the 2 -node bar element and the 3-node triangle are also given.

### 6.4.1 Linearization of the Discrete Equations

In the following, we derive expressions for the continuum tangent stiffness matrix $\mathbf{K}^{\text {int }}$. As will be seen, part of the expression can be derived independently of the material response. These expressions are completed upon introduction of the constitutive relation. The continuum rate form of the constitutive relation will be used, i.e., linearization is carried out prior to introduction of the stress-update algorithm. Specific examples for the continuum tangent matrices for hyperelastic materials and elastic-plastic materials are presented in Section 6.4.2.

For notational convenience, we will develop the tangential stiffness matrix by relating rates of the internal nodal forces $\dot{\mathbf{f}}^{i n t}$ to the nodal velocities $\dot{\mathbf{d}}$. Thus the stiffness matrices $\mathbf{K}^{\text {int }}$ can be derived by taking the material time-derivative of the nodal internal forces. The procedure is identical to relating an infinitesmal increment of nodal displacements $d \mathbf{f}^{i n t}$ to an infinitesmal increment of nodal displacements $d \mathbf{d}$, and we will occasionally recast the equations in that form; the dot notation is chosen for convenience. The derivation is perfectly rigorous for
any continuously differentiable residual; for rougher residuals, directional derivatives are needed and are described later.

By (4.9.10-11), the internal nodal forces in the total Lagrangian form are given by,

$$
\begin{equation*}
\mathbf{f}^{i n t}=\int_{\Omega_{\mathrm{o}}} \mathcal{B}_{0}^{T} \mathbf{P} d \Omega_{0}, \quad f_{I i}^{\text {int }}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{j}} P_{j i} d \Omega_{0} \tag{6.4.1}
\end{equation*}
$$

where $\mathbf{P}$ is the nominal stress tensor with components $P_{j i}, N_{I}$ are the nodal shape functions and $\left(\mathcal{B}_{0}^{T}\right)_{j I}=\partial N_{I} / \partial X_{j}$. We have chosen the total Lagrangian form because this leads to the simplest derivation. In the total Lagrangian form, (6.4.1), only the nominal stress is a function of time, i.e. it is the only variable which varies with deformation. In the updated Lagrangian form, (4.5.5) the domain of the element (or body), the spatial derivatives $\partial N_{I} / \partial x_{j}$ and the Cauchy stress depend on the deformation, and hence on time.

Taking the material time-derivative of (6.4.7) gives

$$
\begin{equation*}
\dot{\mathbf{f}}^{i n t}=\int_{\Omega_{0}} \mathcal{B}_{0}^{T} \dot{\mathbf{P}} d \Omega_{0}, \quad \dot{f}_{I i}^{\text {int }}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{j}} \dot{P}_{j i} d \Omega_{0} \tag{6.4.2}
\end{equation*}
$$

since $\mathcal{B}_{0}$ and $d \Omega_{0}$ are independent of the deformation, which varies with time. To obtain the stiffness matrix $\mathbf{K}^{\text {int }}$ it is now necessary to express the stress rate $\dot{\mathbf{P}}$ in terms of nodal velocities. However, constitutive equations are not expressed in terms of $\mathbf{P}$ because this stress rate is not objective. So we work in terms of the material time derivative of the PK2 stress, which we have seen is objective.

The material time derivative of the PK2 stress is then related to the material time derivative of the nominal stress by Box 3.2, which gives $\mathbf{P}=\mathbf{S} \cdot \mathbf{F}^{T}$, so

$$
\begin{equation*}
\dot{\mathbf{P}}=\dot{\mathbf{S}} \cdot \mathbf{F}^{T}+\mathbf{S} \cdot \dot{\mathbf{F}}^{T} \quad \text { or } \quad \dot{P}_{i j}=\dot{S}_{i r} F_{r j}^{T}+S_{i r} \dot{F}_{r j}^{T} \tag{6.4.3}
\end{equation*}
$$

Substituting (6.4.3) into (6.4.2) yields

$$
\begin{equation*}
\dot{f}_{i I}^{i n t}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{j}}\left(\dot{S}_{j r} F_{i r}+S_{j r} \dot{F}_{i r}\right) d \Omega_{0} \text { or } d f_{i I}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{j}}\left(d S_{i r} F_{i r}+S_{j r} d F_{i r}\right) d \Omega_{0} \tag{6.4.4}
\end{equation*}
$$

The above shows that the rate (or increment) of the internal nodal forces consists of two distinct parts:

1. The first term involves the rate of stress $(\dot{\mathbf{S}})$ and thus depends on the material response and leads to what is called the material tangent stiffness matrix which we denote by $\mathbf{K}_{\text {mat }}$. Note that although this
term reflects material response, it changes with deformation since $B_{0}$ depends on $\mathbf{F}$.
2. The second term involves the current state of stress, $\mathbf{S}$ and accounts for rotation of the stress with the motion. This term is called the geometric stiffness because it represents for geometric nonlinearities associated with rotation of the stress. It is also called the initial stress matrix to indicate the role of the existing state of stress. It is denoted by $\mathbf{K}_{g e o}$.

Therefore we write Eq. (6.4.4) as

$$
\begin{equation*}
\dot{\mathbf{f}}^{\text {int }}=\dot{\mathbf{f}}^{\text {mat }}+\dot{\mathbf{f}}^{\text {geo }} \quad \text { or } \quad \dot{f}_{i I}^{\text {int }}=\dot{f}_{i I}^{\text {mat }}+\dot{f}_{i I}^{\text {geo }} \tag{6.4.5}
\end{equation*}
$$

where

$$
\begin{equation*}
\dot{f}_{i I}^{m a t}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X j} F_{i r} \dot{S}_{j r} d \Omega_{0}, \quad \dot{f}_{i I}^{g e o}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X j} S_{j r} \dot{F}_{i r} d \Omega_{0} \tag{6.4.6}
\end{equation*}
$$

To simplify the remaining development, we put the above expression into Voigt form. Voigt form is convenient in developing the material stiffness matrices because the tensor of material coefficients, $C_{i j k e}$, which which relates the stress rate to the strain rate is a fourth order tensor; this tensor cannot be handled by readily standard matrix operations. Therefore, the stiffness matrix is conventionally handled in Voigt notation; other ways of handling the fourth order stiffness matrices are discussed later.

We consider the material and geometric effects on the nodal forces one at a time. Referring to Eq. (??), we can see that with the definition of (??), which is

$$
\begin{equation*}
B_{j r I i}^{0}=\operatorname{sym}_{(j, r)}\left(\frac{\partial N_{I}}{\partial X j} F_{i r}\right) \tag{6.4.7}
\end{equation*}
$$

we can rewrite the material increment in the nodal forces, Eq. (6.4.4), in Voigt notation as

$$
\begin{equation*}
\dot{\mathbf{f}}_{\text {mat }}^{\text {int }}=\int_{\Omega_{0}} \mathbf{B}_{0}^{T}\{\dot{\mathbf{S}}\} d \Omega_{0} \tag{6.4.8}
\end{equation*}
$$

where $\mathbf{S}$ is now a column matrix arranged according to the Voigt kinetic rule, Appendix A. It should be stressed that Eq. (6.4.6) is identical to Eq. (6.4.5). We now consider the consitutive equation in the following rate form

$$
\begin{equation*}
\dot{S}_{i j}=C_{i j k l}^{S} \dot{E}_{k l} \text { or }\{\dot{\mathbf{S}}\}=\mathbf{C}^{S}\{\dot{\mathbf{E}}\} \tag{6.4.9}
\end{equation*}
$$

Recall (4.9.27), which gives the following relation in Voigt notation

$$
\begin{equation*}
\{\dot{\mathbf{E}}\}=\mathbf{B}_{0} \dot{\mathbf{d}} \tag{6.4.10}
\end{equation*}
$$

Substituting Eqs. (6.4.9) and (6.4.10) into Eq. (6.4.8) gives

$$
\begin{equation*}
\dot{\mathbf{f}}_{\text {mat }}^{i n t}=\int_{\Omega_{0}} \mathbf{B}_{0}^{T} \mathbf{C}^{S} \mathbf{B}_{0} d \Omega_{0} \dot{\mathbf{d}} \quad \text { or } d \mathbf{f}_{m a t}^{i n t}=\int_{\Omega_{0}} \mathbf{B}_{0}^{T} \mathbf{C}^{S} \mathbf{B}_{0} d \Omega_{0} d \mathbf{d} \tag{6.4.11}
\end{equation*}
$$

So the material tangent stiffness matrix is given by

$$
\begin{equation*}
\mathbf{K}^{m a t}=\int_{\Omega_{0}} \mathbf{B}_{0}^{T} \mathbf{C}^{S} \mathbf{B}_{0} d \Omega_{0} \quad \text { or } \mathbf{K}_{I J}^{m a t}=\int_{\Omega_{0}} \mathbf{B}_{0 I}^{T} \mathbf{C}^{S} \mathbf{B}_{0 J} d \Omega_{0} \tag{6.4.12}
\end{equation*}
$$

The material tangent stiffness relates the increment (or rate) in internal nodal forces to the increment (or rate) of displacement due to material response, which is reflected in the material response matrix $\mathbf{C}^{S}$.

The geometric effect on the nodal forces is obtained as follows. From the definition $\mathcal{B}_{i I}^{0}=\frac{\partial N_{I}}{\partial X_{i}}$ and Eq. (6.4.4), we can write

$$
\begin{align*}
\dot{f}_{i I}^{g e o} & =\int_{\Omega_{0}}\left(\mathcal{B}_{j I}^{0}\right)^{T} S_{j r} \dot{F}_{i r} d \Omega_{0}=\int_{\Omega_{0}}\left(\mathcal{B}_{j I}^{0}\right)^{T} S_{j r} \mathcal{B}_{r J}^{0} d \Omega_{0} \dot{u}_{i J}  \tag{6.4.13}\\
& =\int_{\Omega_{0}}\left(\mathcal{B}_{j I}^{0}\right)^{T} S_{j r} \mathcal{B}_{r J}^{0} d \Omega_{0} \delta_{i j} \dot{u}_{j J} \tag{6.4.14}
\end{align*}
$$

where in the second step we have used (4.9.7), $\dot{F}_{i r}=\mathcal{B}_{r I}^{0} \dot{u}_{i I}$, and in the third step we have added a dummy unit matrix so that the component indices in $\dot{f}_{i I}^{g e o}$ and $\dot{u}_{i J}$ are not the same. Writing the resulting expression for the geometric stiffness gives

$$
\begin{equation*}
\mathbf{f}_{I}=\mathbf{K}_{I J}^{g e o} \dot{\mathbf{u}}_{J} \text { where } \mathbf{K}_{I J}^{g e o}=\int_{\Omega_{0}} \mathrm{~B}_{0 I}^{T} \mathbf{S} \mathrm{~B}_{0 J} d \Omega_{0} \mathbf{I} \tag{6.4.15}
\end{equation*}
$$

Note that the PK2 stress in the above is a square matrix. Each submatrix of the geometric stiffness matrix is a unit matrix; therefore, it follows that the geometric stiffness matrix, like the unit matrix, is invariant with rotation, i.e.

$$
\begin{equation*}
\hat{\mathbf{K}}_{I J}^{\text {geo }}=\mathbf{K}_{I J}^{g e o} \tag{6.4.16}
\end{equation*}
$$

where $\hat{\mathbf{K}}_{I J}^{\text {geo }}$ relates nodal forces to nodal velocities expressed in any alternate set of Cartesian coordinates.

To summarize

$$
\begin{equation*}
d \mathbf{f}^{i n t}=\mathbf{K}^{i n t} d \mathbf{d} \quad \text { or } \dot{\mathbf{f}}^{i n t}=\mathbf{K}^{i n t} \mathbf{d} \quad \text { where } \quad \mathbf{K}^{i n t}=\mathbf{K}^{m a t}+\mathbf{K}^{g e o} \tag{6.4.17}
\end{equation*}
$$

where the material tangent stiffness and the geometric stiffness are given by Eqs. (6.4.12) and 6.4.15), respectively. The material tangent stiffness reflects the effect on the nodal internal forces of the deformation of the material. The geometric stiffness reflects the effects of the rotation and deformation on the current state of stress.

The above forms are easily converted to updated Lagrangian forms by letting the current configuration be a reference configuration, as in Section 4.??. From Eqs. (4.9.29), we recall that taking the current configuration as the reference configuration gives

$$
\begin{equation*}
\mathbf{B}_{0} \rightarrow \mathbf{B} \quad \mathrm{~B}_{0} \rightarrow \mathrm{~B} \quad \mathbf{S} \rightarrow \sigma \quad d \Omega_{0} \rightarrow d \Omega \tag{6.4.18}
\end{equation*}
$$

Also, referring to Section 4.??, we note that when a fixed current configuration becomes the reference configuration, then

$$
\begin{equation*}
\mathbf{F} \rightarrow \mathbf{I} \tag{6.4.19}
\end{equation*}
$$

In Section (???), we have seen that the relationship rate of the PK2 stress to the given strain is equivalent to that of the Truesdell rate of the Cauchy stress to the rate-of-deformation in the current configuration, so

$$
\begin{equation*}
\mathbf{C}^{S} \rightarrow \mathbf{C}^{\sigma \mathcal{T}} \tag{6.4.20}
\end{equation*}
$$

Thus, Eqs. (6.4.13) and (6.4.16) become

$$
\begin{align*}
& \mathbf{K}_{I J}^{m a t}=\int_{\Omega} \mathbf{B}_{I}^{T} \mathbf{C}^{\sigma \tau} \mathbf{B}_{J} d \Omega \quad \mathbf{K}^{m a t}=\int_{\Omega} \mathbf{B}^{T} \mathbf{C}^{\sigma \tau} \mathbf{B} d \Omega \\
& \mathbf{K}_{I J}^{\text {geo }}=\mathbf{I} \int_{\Omega} \mathrm{B}_{I}^{T} \sigma \mathrm{~B}_{J} d \Omega \tag{6.4.21}
\end{align*}
$$

These forms are generally easier to use than the total Lagrangian forms, since $\mathbf{B}$ is more easily constructed than $\mathcal{B}_{0}$ and many material laws are developed in terms of Cauchy stress. It is not possible to write a convenient expression for the entire geometric stiffness matrix in this notation. Note that either the material or geometric stiffness can be used in total Lagrangian form with the other in updated Lagrangian form. The numerical values of the matrices in total and updated lagrangian form are identical, and the choice is a matter of convenience.

The integrand in the geometric stiffness is a scalar for given values of $I$ and $J$, so Eq. (6.4.21) can be written as

$$
\begin{equation*}
\mathbf{K}_{I J}^{g e o}=\mathbf{I} H_{I J} \quad \text { where } \quad H_{I J}=\int_{\Omega} \mathrm{B}_{I}^{T} \mathbf{S B}_{J} d \Omega \tag{6.4.22}
\end{equation*}
$$

Alternate Derivations. In this Section the tangent stiffness matrix is derived in terms of the convected rate of the Kirchhoff stress. Many of the relations in nonlinear mechanics take on a particular elegance and simplicity when expressed in terms of the Kirchhoff stress. In addition, the following development relies
more on indicial notation and the shift to Voigt notation is not made until the last steps.

Noting that the Kirchhoff stress $\tau$ is related to the nominal stress by (???), $\mathbf{P}=\mathbf{F}^{-1} \cdot \tau$, the rate form of the relation between the nominal stress and the Kirchhoff stress is obtained by taking the material time derivative

$$
\begin{equation*}
\dot{\mathbf{P}}=\mathbf{F}^{-1} \cdot \dot{\tau+} \dot{\mathbf{F}}^{-1} \cdot \tau \tag{6.4.24}
\end{equation*}
$$

Using the result $\frac{D\left(\mathbf{F}^{-1} \cdot \mathbf{F}\right)}{D t}=0$, it is straightforward to show that

$$
\begin{equation*}
\left(\mathbf{F}^{-1}\right)=-\mathbf{F}^{-1} \cdot \dot{\mathbf{F}} \cdot\left(\mathbf{F}^{-1}\right)=-\mathbf{F}^{-1} \cdot \mathbf{L} \tag{6.4.25}
\end{equation*}
$$

where the second relation follows from (???). Thus the expression (6.4.24) for the nominal stress rate is written as

$$
\begin{equation*}
\dot{\mathbf{P}}=\mathbf{F}^{-1}(\dot{\tau}-\mathbf{L} \cdot \tau) \tag{6.4.26}
\end{equation*}
$$

Using (5. ????) to relate the material rate of theKirchhoff stress to its convected rate, $\tau^{\nabla_{c}}=\dot{\tau}-\mathbf{L} \cdot \tau-\tau \cdot \mathbf{L}^{T}$, (6.4.26) is written as

$$
\begin{equation*}
\dot{\mathbf{P}}=\mathbf{F}^{-1}\left(\dot{\tau}^{\nabla_{c}}+\tau \cdot \mathbf{L}^{T}\right) \tag{6.4.27}
\end{equation*}
$$

Writing (6.4.27) in indicial notation, we obtain

$$
\begin{equation*}
\dot{P}_{j i}=F^{-1}\left(\tau_{k i}^{\nabla_{c}}+\tau_{k l} L_{i l}\right)=\frac{\partial X_{j}}{\partial x_{k}}\left(\tau_{k i}^{\nabla_{c}}+\tau_{k l} L_{i l}\right) \tag{6.4.28}
\end{equation*}
$$

Substituitng the above into (6.4.2) gives

$$
\begin{array}{r}
\dot{f}_{i I}^{i n t}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial X_{j}} \frac{\partial X_{j}}{\partial x_{k}}\left(\tau_{k i}^{\nabla_{c}}+\tau_{k l} L_{i l}\right) d \Omega_{0} \\
=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial x_{k}}\left(\tau_{k i}^{\nabla_{c}}+\tau_{k l} L_{i l}\right) d \Omega_{0} \\
=\int_{\Omega_{0}} N_{I, k}\left(\tau_{k i}^{\nabla_{c}}+\tau_{k l} L_{i l}\right) d \Omega_{0} \tag{6.4.29}
\end{array}
$$

where the second expression follows from the first by the chain rule; in the third expression we have used the notation $N_{I, k}=\partial N_{I} / \partial x_{k}$. This is the counterpart of (6.4.4) in terms of the Kirchhoff stress.

This result can easily be transformed to an updated Lagrangian form with the integral over the current domain. Using (3.2.18, $d \Omega=J d \Omega_{0}$ and the relation (5.???) between the convected rate of Kirchhoff stress and the Truesdell rate of Cauchy stress ( $\tau^{\nabla_{c}}=J \sigma^{\nabla_{T}}$ ) the expression (6.4.29) yields

$$
\begin{equation*}
\dot{f}_{i I}=\int_{\Omega} N_{I, k}\left(\sigma_{k i}^{\nabla_{T}}+\sigma_{k l} L_{i l}\right) d \Omega \tag{6.4.30}
\end{equation*}
$$

which is the updated Lagrangian counterpart of Eq. (6.4.4); (6.4.30) could also be obtained by making the current configuration the reference configuration EXERCISE

An alternative derivation of (6.4.29) is given as follows. Recall Eq. (6.4.3):

$$
\begin{equation*}
\dot{\mathbf{P}}=\dot{\mathbf{S}} \cdot \mathbf{F}^{T}+\mathbf{S} \cdot \mathbf{F}^{T} \tag{6.4.33}
\end{equation*}
$$

Now note that this relation can be written as

$$
\begin{equation*}
\dot{\mathbf{P}}=\mathbf{F}^{-1} \cdot \mathbf{F} \cdot \dot{\mathbf{S}} \cdot \mathbf{F}^{T}+\mathbf{F}^{-1} \cdot \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{T} \cdot \mathbf{F}^{-T} \cdot \dot{\mathbf{F}}^{T} \tag{6.3.34}
\end{equation*}
$$

Using the push-forward relation (5????) for the convected rate of Kirchhoff stress in terms of the rate of the PK2 stress, $\tau^{\nabla_{c}}=\mathbf{F} \cdot \dot{\mathbf{S}}^{\cdot} \cdot \mathbf{F}^{T}$, and (3.3.18), $\mathbf{L}^{T}=\mathbf{F}^{-T} \cdot \dot{\mathbf{F}}^{T}$, (6.4.34) can be written as

$$
\begin{equation*}
\dot{\mathbf{P}}=\mathbf{F}^{-1} \cdot\left(\tau^{\nabla_{c}}+\tau \cdot \mathbf{L}^{T}\right) \tag{6.4.35}
\end{equation*}
$$

which is the same as (6.4.27). The tangent stiffness expression (6.4.29) follows in an identical fashion.

To complete the derivation of the material tangent stiffness matrix (6.4.29), it is necessary to introduce the constitutive relation to relate the convected stress rate to the nodal velocities. We write the constitutive relation (rate-independent material response) in the form

$$
\begin{equation*}
\tau_{i j}^{\nabla_{c}}=C_{i j k l}^{\tau} D_{k l} \tag{6.4.38}
\end{equation*}
$$

where the superscript $\tau$ on the tangent modulus $C_{i j k l}^{\tau}$ indicates that it relates the Kirchoff stress rate to the rate-of-deformation. This tangent modulus possesses the minor symmetries, i.e., $C_{i j k l}^{\tau}=C_{j i k l}^{\tau}=C_{i j k}^{\tau}$. For some materials, this tangent modulus tensor also has major symmetry, i.e, $C_{i j k l}^{\tau}=C_{k i j}^{\tau}$, i.e. for hyperelastic materials and for associated rate-independent plasticity based on the Kirchhoff stress; see Chapter 5. This tangent modulus tensor for non-associated rateindependent plasticity is not symmetric. We will also show in the following that the tangent modulus for associated plasticity based on the Jaumann rate of the Cauchy stress also does not have major symmetry.

An expression for the material tangent stiffness matrix is now derived using the general form (6.4.38) of the constitutive relation for a rate-independent material. Specific examples of this relation and the associated tangent moduli are given at the end of this subsection. In this derivation, instead of immediately changing to Voigt notation, we will continue with indicial notation to the final expression and then translate that to Voigt notation.

Substituting (6.4.38) into (6.4.32) gives

$$
\begin{equation*}
K_{I J i j}^{m a t} d_{J j}=\int_{\Omega_{0}} \frac{\partial N_{I}}{\partial x_{k}} C_{k i j l}^{\tau} D_{j l} d \Omega_{\mathrm{o}} \tag{6.4.39}
\end{equation*}
$$

Recall from (4.4.7b), that the rate of deformation tensor is the symmetric part of the spatial velocity gradient, $D_{k \ell}=v_{(k, \ell)}=\operatorname{sym}\left(v_{k I} N_{I, \ell}\right)$. Substituting this and the rate form of the constitutive equation (6.4.38) into (6.4.39) we obtain

$$
\begin{align*}
K_{I J i j}^{t a n} \dot{d}_{J j} & =\int_{\Omega_{0}} N_{I, k} C_{k i j l}^{\tau} v_{j, l} d \Omega_{0} \\
& =\int_{\Omega_{0}} N_{I, k} C_{k i j l}^{\tau} N_{J, l} \dot{u}_{j J} d \Omega_{\mathrm{o}} \\
& =\int_{\Omega_{0}} N_{I, k} C_{k i j l}^{\tau} N_{J, l} d \Omega_{0} \dot{u}_{j J} \tag{6.4.41}
\end{align*}
$$

where in the second of the above we have used the result $C_{k i j l}^{\tau} D_{j l}=C_{k i j l}^{\tau} v_{j l}$ which follows the minor symmetry of the tangent modulus matrix, $C_{k i j l}^{\tau}=C_{k i j l}^{\tau}$. From (6.4.41), the material tangent stiffness matrix is defined by

$$
\begin{equation*}
K_{I J i j}^{m a t}=\int_{\Omega_{0}} N_{I, k} C_{k i j l}^{\tau} N_{J, l} d \Omega_{0} \tag{6.4.42}
\end{equation*}
$$

This expression can also be written as an integral over the current domain, i.e.,

$$
\begin{equation*}
K_{I J i j}^{m a t}=\int_{\Omega_{0}} N_{I, k} \frac{1}{J} C_{k j j}^{\tau} N_{J, l} d \Omega=\int_{\Omega} N_{I, k} C_{k i j l}^{\sigma \mathcal{T}} N_{J, l} d \Omega \tag{6.4.43}
\end{equation*}
$$

where we have used (????) to write the second expression.
We now convert the above to Voigt notation. Equation (6.4.43) is now written as

$$
\begin{equation*}
K_{I J r s}^{m a t}=\int_{\Omega} N_{A, k} \delta_{r i} C_{k i j l}^{\sigma \tau} N_{B, l} \delta_{s j} d \Omega \tag{6.4.46}
\end{equation*}
$$

Noting that $C_{k i j l}^{\tau}=C_{i k j l}^{\tau}=C_{k i l j}^{\tau}$, (6.4.46) and using (4.5.19b), the above can be written as

$$
\begin{equation*}
K_{I J r s}^{m a t}=\int_{\Omega} B_{i k A r} \frac{1}{J} C_{k i j l}^{\tau} B_{j l B s} d \Omega \tag{6.4.47}
\end{equation*}
$$

which is given in matrix form as

$$
\begin{equation*}
\mathbf{K}_{I J}^{m a t}=\int_{\Omega} \mathbf{B}_{I}^{T} C^{\sigma \tau} \mathbf{B}_{J} d \Omega=\int_{\Omega} J^{-1} \mathbf{B}_{I}^{T} C^{\tau} \mathbf{B}_{J} d \Omega \tag{6.4.49}
\end{equation*}
$$

This form is identical to the linear stiffness matrix except that the material response matrix $J^{-1} \mathbf{C}^{\tau}$ relates the convected rate of Kirchhoff stress to the rate-of-deformation (or alternatively the response matrix $\mathbf{C}^{\boldsymbol{T}}$ relates the Truesdell rate of Cauchy stress to the rate-of-deformation).

Some examples of tangent moduli for different materials are given in the following. Detailed derivations of the tangent material stiffness matrices for specific finite elements are given in Section 6.4.2 below.

Tangent Modulus for Hyperelastic Material The rate form of the constitutive relation for a hyperelastic material is given by (5.x), i.e,

$$
\begin{equation*}
\tau^{\nabla_{c}}=C_{i j k l} D_{k l} \quad \text { or } \quad \sigma^{\nabla \mathcal{T}}=C_{i j k l} D_{k l} \tag{6.4.50}
\end{equation*}
$$

Thus from (6.4.50), the tangent modulus for a hyperelastic material is given by

$$
\begin{equation*}
C_{i j k l}^{\tau}=J C_{i j k l}^{\sigma \mathcal{T}}=F_{i m} F_{j n} F_{k p} F_{l q} C_{m n p q}^{S E} \tag{6.4.51}
\end{equation*}
$$

where from (5.y) $\bar{C}_{m n p q}$ is derived from the hyperelastic potential, i.e.,

$$
\begin{equation*}
C_{m n p q}^{S E}=\frac{\partial S_{m n}}{\partial E_{p q}}=2 \frac{\partial S_{m n}}{\partial C_{p q}}=2 \frac{\partial^{2} W(C)}{\partial C_{m n} \partial C_{p q}} \tag{6.4.52}
\end{equation*}
$$

An interesting feature of the expression (6.4.51) is that for a hyperelastic material the rate form of the material response is expressed naturally in terms of the convected rate of Kirchhoff stress. The expression (6.4.51) contains no geometric terms consisting of the product of the current stress the spatial velocity gradient (or its symmetric or antisymmetric parts)??????MORAN???. In many materials, (including the elastic-plastic material considered in the following) the tangent moduli are functions of the initial stresses.

Tangent Modulus for Hypoelastic-Plastic Material We will now develop the tangent modulus for hypoelastic-plastic materials based on i) the Kirchhoff stress and ii) the Cauchy stress. The elastic response is assumed to be given by relating the Jaumann rate of stress to the elastic part of the rate of deformation tensor.
i) Kirchhoff Stress Formulation Recalling the relation ( ) which relates the Jaumann rate of Kirchhoff stress to the rate of deformation tensor, we have

$$
\begin{equation*}
\tau_{i j}^{\nabla c}=C_{i j k l}^{\tau} D_{k l} \tag{6.4.53}
\end{equation*}
$$

where $C_{i j k l}^{\tau}$ is the elastic-plastic tangent modulus given in (5.??) Using the relationship (5.???) between the convected rate and the Jaumann rate gives

$$
\begin{align*}
\tau_{i j}^{\nabla c} & =\tau_{i j}^{\nabla \boldsymbol{g}}-D_{i k} \tau_{k j}-\tau_{i k} D_{k j} \\
& =C_{i j k l}^{e p} D_{k l}-D_{i k} \tau_{k j}-\tau_{i k} D_{k j} \\
& =\left(C_{i j k l}^{e p}-\delta_{i l} \tau_{k j}-\tau_{i k} \delta_{j l}\right) D_{k l} \tag{6.4.54}
\end{align*}
$$

The terms involving the stress tensor are a result of the use of the Jaumann rate in the hypoelastic relation and the difference between the Jaumann rate and the convected rate which appears in the expression for the tangent stiffness matrix. These are traditionally interpreted as part of the material tangent stiffness matrix although they can also be regarded as geometric terms due to the convection of the stress.

Because of the symmetry of $D_{k l}$, the last expression can be written as

$$
\begin{align*}
\tau_{i j}^{\nabla_{c}} & =\left(C_{i j k l}^{\tau}-\frac{1}{2}\left(\delta_{i l} \tau_{k j}+\tau_{i k} \delta_{j l}+\delta_{i k} \tau_{l j}+\tau_{i l} \delta_{j k}\right)\right) D_{k l} \\
& =C_{i j k l}^{t a n} D_{k l} \tag{6.4.43}
\end{align*}
$$

where

$$
\begin{equation*}
C_{i j k l}^{t a n}=C_{i j k l}^{e p}-\frac{1}{2}\left(\delta_{i l} \tau_{k j}+\tau_{i k} \delta_{j l}+\delta_{i k} \tau_{l j}+\tau_{i l} \delta_{j k}\right) \tag{6.4.55}
\end{equation*}
$$

is the tangent modulus. Note that this tangent modulus has major and minor symmetries for an associated law, so the tangent stiffness is symmetric.
ii) Cauchy Stress Formulation We will now develop the tangent stiffness for a hypoelastic-plastic material based on the Cauchy stress and we will show that the resulting stiffness is not symmetric. The constitutive relation in elastoplasticity relates the Jaumann rate of the Cauchy stress to the rate-of-deformation:

$$
\begin{equation*}
\sigma_{i j}^{\nabla \mathfrak{J}}=C_{i j k l}^{\sigma \mathfrak{J}} D_{k l} \tag{6.4.56}
\end{equation*}
$$

where $C_{i j k l}^{\sigma y}$ is the elastic-plastic tangent modulus. The Jaumann rate is used because the invariants of the Cauchy stress tensor remain constant when the Jaumann rate vanishes, see section 5.??. Using the relationship (5.??) between the Jaumann rates of Kirchhoff and Cauchy stresses, the convected rate is written as

$$
\tau_{i j}^{\nabla_{c}}=\tau_{i j}^{\nabla_{J}}-D_{i k} \tau_{k j}-\tau_{i k} D_{k j}
$$

$$
\begin{equation*}
=J \sigma_{i j}^{\nabla}+D_{k k} \tau_{i j}-D_{i k} \tau_{k j}-\tau_{i k} D_{k j} \tag{6.4.57}
\end{equation*}
$$

## Linearization with Directional Derivatives

Three difficulties arise in applying the traditional Newton-Raphson method to solid mechanics problems:

1. the nodal forces are not continuously differentiable functions of the nodal displacements for materials such as elastic-plastic materials;
2. for path-dependent materials, the classical Newton method pollutes the constitutive models since the intermediate solutions to the linear problem in the iterative procedure, $\mathbf{d}_{v}$, are not part of the actual load path;
3. for large incremental rotations and deformation, the linearized increments introduce a substantial error

In order to overcome these difficulties, the Newton-Raphson method is often modified as follows:

1. directional derivatives, also called Frechet derivatives, are used to develop the tangent stiffness;
2. a secant method is used instead of a tangent method and the last converged solution is used as the iteration point.
3. formulas depending on increment size are used to relate the increments forces and displacements.

It should be pointed out that the third difficulty only arises when the secant method is used to circumvent the second difficulty. If a tangent Newton method is used for a smooth material, there is no advantage to carrying higher order terms in the geometric terms.

To illustrate the need for directional derivatives with elastic-plastic materials, consider the following example. The two-bar truss shown in Fig. 6.??? has been loaded so that the stresses in both bars are compressive and equal, and both bars are at the compressive yield stress. For simplicity, we consider only material nonlinearities and neglect geometric nonlinearities. If an arbitrary load increment $\Delta \mathbf{f}_{1}^{\text {ext }}$ is now applied to node 1 , the tangent stiffness matrix will depend on the incremental displacement $\Delta \mathbf{u}_{1}$ because the derivatives of the internal nodal displacements depend on the direction of the displacement increment. The residual is not a continuoiusly differentiable function of the incremental nodal displacments at this point, because the change in nodal internal forces depends on whether the response of the either rod is elastic or plastic. In this case, there are four lines of discontinuity for the derivatives, as shown in Figure ???. These result from the fact that if the displacment increment results in a tensile strain increment, then the bar unloads elastically, so the tangent modulus changes from the elastic modulus $E$ to the plastic modulus $H_{p}$.

The internal nodal forces in the current configuration are

$$
\left\{\begin{array}{l}
f_{x 1}  \tag{6.4.58}\\
f_{y 1}
\end{array}\right\}^{\text {int }}=A\left\{\begin{array}{l}
\sigma_{0} \cos \theta-\sigma_{0} \cos \theta \\
\sigma_{0} \sin \theta+\sigma_{0} \sin \theta
\end{array}\right\}
$$

where $\sigma_{0}$ is the current yield stress; the above is obtained by assembling the internal nodal forces for rod elements as given by Eq. (E.4.6.7). For each rod element, there are two possibilities depending on the direction of the force: either the rod continues to load witha plastic modulus, or it unloads with an elastic modulus. As a result, the tangent stiffness in this configuration can take on four different values.


Figure ??. A two-bar truss in a state with both bars in compressive yield and the four quadrants of directional derivatives.

The nodal force $f_{1 x}$ is shown as a function of the two components of the nodal displacement increment in Figure ???, where the discontinuity in the derivatives is clearly apparent. Obviously, a standard derivative cannot be evaluated since it has four different values.

These regimes of the four different responses are illustrated in Fig. 6.??, which shows the four quadrants in the space of the components of the nodal
displacement. The tangent stiffness for dkisplacement increments in the four quadrants is given by the following:
in quadrant 1 :

$$
\mathbf{K}^{i n t}=\frac{A E}{\ell}\left[\begin{array}{cc}
2 \cos ^{2} \theta & 0  \tag{6.4.59a}\\
0 & 2 \sin ^{2} \theta
\end{array}\right]
$$

in quadrant 2 :

$$
\mathbf{K}^{\text {int }}=\frac{A}{\ell}\left[\begin{array}{cc}
\left(E+H_{p}\right) \cos ^{2} \theta & \left(E-H_{p}\right) \sin \theta \cos \theta  \tag{6.5.59b}\\
\left(E-H_{p}\right) \sin \theta \cos \theta & \left(E+H_{p}\right) \sin ^{2} \theta
\end{array}\right]
$$

in quadrant 3 :

$$
\mathbf{K}^{\text {int }}=\frac{A H_{p}}{\ell}\left[\begin{array}{cc}
2 \cos ^{2} \theta & 0  \tag{6.5.59c}\\
0 & 2 \sin ^{2} \theta
\end{array}\right]
$$

in quadrant 4:

$$
\mathbf{K}^{\text {int }}=\frac{A}{\ell}\left[\begin{array}{cc}
\left(E+H_{p}\right) \cos ^{2} \theta & \left(H_{p}-E\right) \sin \theta \cos \theta  \tag{6.5.59d}\\
\left(H_{p}-E\right) \sin \theta \cos \theta & \left(E+H_{p}\right) \sin ^{2} \theta
\end{array}\right]
$$



Figure 6.4. Schematic of potential energy, a stable equilibrium solution, and the contours for the potential with their gradient -r.

To deal with this type of behavior in a methodical manner, a Frechet derivative, often called a directional derivative, must be used. The Frechet derivative is defined by

$$
\begin{equation*}
\frac{d f(\mathbf{d})}{d \mathbf{d}_{[\Delta \mathbf{d}]}}=\left.\lim _{\varepsilon \rightarrow 0} \frac{d}{d \varepsilon} f(\mathbf{d}+\varepsilon \Delta \mathbf{d})\right|_{\varepsilon=0} \tag{6.4.60}
\end{equation*}
$$

The subcript on the lower term gives the direction in which the derivative is taken. The notation $D f(\mathbf{d}) \cdot[\Delta \mathbf{d}]$ is often used for the directional derivative in the finite element literature..

The value of directional derivative depends on the direction of the increment of the independent variable. The tangent stiffnesses in (??) are based on directional derivatives for the nodal forces have been given in (???).

In the computation of the tangent stiffness matrix, and in particular the material tangent stiffness, directional derivatives are used for elastic-plastic materials. The direction is based on the last displacement increment in the iterative procedure. If the load increment suddenly reverses, the last increment is not in the direction of the next solution increment, and the directional derivative may be quite erroneous,. However, after several iterations, the correct direction is determined for the displacement increment and the directional derivative gives the correct rate of change of the nodal forces.

Since the directional derivative to a specific value of the nodal displacements, this approach cannot be used with the standard tangent Newton described in Box ???. Instead, a secant Newton method must be used. The secant Newton method is given in Box ???.

External Load Stiffness. An important class of loads are follower loads, which change with the configuration of the body. Examples of follower forces are shown in Figure ??. Pressure loading is a common example of a follower load. Since a pressure loading is always normal to the surface, as the surface moves, the nodal external forces change even if the pressure is constant. These effects are accounted for in the Jacobian matrix $\mathbf{K}^{\text {ext }}$, which is also called the load stiffness.

The load stiffness $\mathbf{K}^{\text {ext }}$ is obtained by relatinng the time derivative (or increment) of the external nodal forces to the time derivative (or increment) of nodal displacements. We first consider loading by pressure, $p(\mathbf{x}, t)$. The external nodal forces on a surface of element $e$ are given by letting $\mathbf{t}=-p \mathbf{n}$ in Eq. (4.9.13):

$$
\begin{equation*}
\mathbf{f}_{I}^{e x t}=-\int_{\Gamma} N_{I} p \mathbf{n} d \Gamma \tag{6.4.61}
\end{equation*}
$$

Let the surface $\Gamma$ be described in terms of two variables $\xi$ and $\eta$. For a quadrilateral surface element, these independent variables are the parent element coordinates on the biunit square. As in Eq. (E4.3.1b), since $\mathbf{n} d \Gamma=\mathbf{x}, \xi \times \mathbf{x},{ }_{\eta} d \xi d \eta$ becomes

$$
\begin{equation*}
\mathbf{f}_{I}^{e x t}=\int_{-1-1}^{1} \int_{I}^{1} p N_{I} \mathbf{x}, \xi \times{ }_{\xi} \times \mathbf{x}_{\eta} d \xi d \eta \tag{6.4.62}
\end{equation*}
$$

Taking the time derivative of the above gives

$$
\begin{equation*}
\dot{f}_{I}^{e x t}=\int_{-1-1}^{1} \int_{I}^{1} N_{I}\left(\dot{p} \mathbf{x}_{\xi} \times \mathbf{x}_{\eta}+p \mathbf{v}, \xi \times \mathbf{x},{ }_{\eta}+p \mathbf{x}, \xi \times \mathbf{v}_{\eta}\right) d \xi d \eta \tag{6.4.63}
\end{equation*}
$$

The first term is the rate of change of the external forces due to the rate of change of the pressure. In many problems the rate of change of pressure is prescribed as part of the problem. In some problems, such as in fluid-structure interaction problems, the pressure may arise from changes of the geometry; these effects must then be linearized and added to the load stiffness. The second two terms represent the changes in the external nodal forces due to the change in the direction of the surface and the area of the surface. These are the terms which are reflected in the external load stiffness, so

$$
\begin{equation*}
\mathbf{K}_{I K}^{e x t} \mathbf{v}_{K}=\int_{-1-1}^{1} \int_{I}^{1} p N_{I}\left(\mathbf{v}, \xi \times \mathbf{x},{ }_{\eta}+\mathbf{x}, \xi \times \mathbf{v}_{\eta}\right) d \xi d \eta \tag{6.4.64}
\end{equation*}
$$

At this point, it is convenient to switch partially to indicial notation. Taking the dot product of the above with the unit vector $\mathbf{e}_{i}$ gives

$$
\begin{align*}
\mathbf{e}_{i} \cdot \mathbf{K}_{I K}^{e e^{\prime} t} \mathbf{v}_{K} & \equiv K_{i j I K}^{e x t} v_{j K} \\
& \equiv \int_{-1-1}^{1} \int_{I}^{1} p N_{I}\left[N_{J, \xi} \mathbf{e}_{i} \cdot\left(\mathbf{e}_{k} \times \mathbf{x}_{\eta}\right)+N_{J, \eta} \mathbf{e}_{i} \cdot\left(\mathbf{x}_{, \xi} \times \mathbf{e}_{k}\right)\right] d \xi d \eta \tag{6.4.65}
\end{align*}
$$

where we have expanded the velocity field in terms of the shape functions by $\mathbf{v}, \xi=\mathbf{v}_{K} N_{K, \xi}$. We now define

$$
\begin{equation*}
H_{i k}^{\eta} \equiv e_{i k \ell} x_{\ell, \eta} \quad H_{i k}^{\xi}=e_{i k} x_{\ell, \xi} \tag{6.4.66}
\end{equation*}
$$

Using these definitions and Eq. (6.4.65), we obtain

$$
\begin{equation*}
K_{i j I J}^{e x t}=\int_{-1-1}^{1} \int_{I}^{1} p N_{I}\left(N_{J, \xi} H_{i j}^{\eta}-N_{J, \eta} H_{i j}^{\xi}\right) d \xi d \eta \tag{6.4.67}
\end{equation*}
$$

or

$$
\mathbf{K}_{I J}^{e x t}=\int_{-1-1}^{1} \int_{I}^{1} p N_{I}\left(N_{J, \xi} \mathbf{H}^{\eta}-N_{J, \eta} \mathbf{H}^{\xi}\right) d \xi d \eta
$$

If we write out the matrices $\mathbf{H}^{\xi}$ and $\mathbf{H}^{\eta}$ we have

$$
\begin{align*}
& \mathbf{K}_{I J}^{e x t}=\int_{-1-1}^{1} \int_{I}^{1} p N_{I}\left(N_{J, \xi}\left[\begin{array}{ccc}
0 & z_{\eta} & -y_{\eta} \\
-z, \eta & 0 & x_{,} \\
y_{\eta} & -x, \eta & 0
\end{array}\right]\right. \\
&\left.-N_{J, \eta}\left[\begin{array}{ccc}
0 & z, \xi & -y_{, \xi} \\
-z, \xi & 0 & x, \xi \\
y, \xi & -x, \xi & 0
\end{array}\right]\right) d \xi d \eta \tag{6.4.68}
\end{align*}
$$

which is the load stiffness of any surface which is generated from a biunit square in the parent element loaded by a pressure $p$. The load surface for a surface originating can be similarly expressed in terms of the area coordinates, although the limits of integration need to be changed. This load stiffness reflects the effect of the change in geometry on the nodal forces: both alterations in the direction of the loaded surfaces and size of the surface will changes in the nodal forces. It is immediately apparent from (6.4.68) that the submatrices of the load stiffness matrix are not symmetric, so the complete matrix is not symmetric. However, it can be shown that for a closed structure in a constant pressure field, the assembled external load stiffness is symmetric.

## Example 6.1. Three-Node Triangle Element.

We first consider the three-node triangle element in two dimensions as in Example 4.1. The tangent stiffness matrix is derived and explicit forms for hyperelastic and rate-independent hypoelastic-plastic materials are given. The geometric tangent stiffness matrix, which is independent of material response, is then derived. Finally, the external load matrices are derived for a pressure load along any edge.

Material Tangent Stiffness Matrix. We consider the case of plane strain deformation (using the $x-y$ plane). The only velocity components are $v_{x}$ and $v_{y}$ and derivatives with respect to $z$ vanish. The tangent stiffness matrix for a rateindependent material given by Eq. (6.3.36):

$$
\begin{equation*}
K^{t a n}=\int \mathbf{B}^{T}[C] \mathbf{B} d A \tag{E6.1.1}
\end{equation*}
$$

where $A$ is the current area of the element and we have assumed a unit thickness (see Eq. (4.??)).

$$
\left[C_{a b}^{t a n}\right]=\left[\begin{array}{ccc}
C_{1111}^{\sigma \mathcal{T}} & C_{1122}^{\sigma \mathcal{T}} & C_{1112}^{\sigma \mathcal{T}}  \tag{E6.1.2}\\
C_{2211}^{\sigma \mathcal{T}} & C_{2222}^{\sigma \mathcal{T}} & C_{2212}^{\sigma \boldsymbol{T}} \\
C_{1211}^{\sigma \mathcal{T}} & C_{1222}^{\sigma \mathcal{T}} & C_{1212}^{\sigma \sigma}
\end{array}\right]
$$

The $\mathbf{B}$ matrix is given by Eq. (E4.1.45):

$$
\mathbf{B}=\frac{1}{2 A}\left[\begin{array}{cccccc}
y_{23} & 0 & y_{31} & 0 & y_{12} & 0  \tag{E6.1.3}\\
0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\
x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12}
\end{array}\right]
$$

The material tangent stiffness matrix, Eq. (6.4.81), is rewritten, using Eqs. (6.4.82-6.4.84) as

$$
\left.\left.\begin{array}{rl}
\mathbf{K}^{\tan }=\int_{A_{0}}\left(\frac{1}{2 A}\right)^{2}\left[\begin{array}{ccc}
y_{23} & 0 & x_{32} \\
0 & x_{32} & y_{23} \\
y_{31} & 0 & x_{13} \\
0 & x_{13} & y_{31} \\
y_{12} & 0 & x_{21} \\
0 & x_{21} & y_{12}
\end{array}\right]\left[\begin{array}{ccc}
C_{1111}^{\sigma \mathcal{T}} & C_{1122}^{\sigma \boldsymbol{T}} & C_{1112}^{\boldsymbol{\sigma} \tau} \\
C_{2211}^{\sigma \mathcal{T}} & C_{2222}^{\sigma \mathcal{T}} & C_{2212}^{\sigma \mathcal{T}} \\
C_{1211}^{\sigma \mathcal{T}} & C_{1222}^{\sigma \mathcal{T}} & C_{1212}^{\sigma \tau}
\end{array}\right]  \tag{E6.1.4}\\
& {\left[\begin{array}{ccccc}
y_{23} & 0 & y_{31} & 0 & y_{12} \\
0 & x_{32} & 0 & x_{13} & 0 \\
y_{21} \\
x_{32} & y_{23} & x_{13} & y_{31} & x_{21}
\end{array} y_{12}\right.}
\end{array}\right] d A_{0}\right]
$$

Assuming the integrand to be constant, we obtain, by multiplying the integrand by the element area $A_{0}$ (note that, for plane strain, a unit thickness is assumed and the element volume is given by $\left.V_{0}=A_{0}(1)=A_{0}\right)$.

$$
\begin{align*}
& K_{A B}^{t a n}= \frac{A_{0}}{4 A^{2}}\left[\begin{array}{ccc}
y_{23} & 0 & x_{32} \\
0 & x_{32} & y_{23} \\
y_{31} & 0 & x_{13} \\
0 & x_{13} & y_{31} \\
y_{12} & 0 & x_{21} \\
0 & x_{21} & y_{12}
\end{array}\right]\left[\begin{array}{llll}
C_{1111}^{\sigma \tau} & C_{122}^{\sigma \tau} & C_{1112}^{\sigma \tau} \\
C_{2211}^{\sigma} \tau & C_{2222}^{\sigma \tau} & C_{2212}^{\sigma \tau} \\
C_{1211}^{\sigma \tau} & C_{1222}^{\sigma} \boldsymbol{\tau} & C_{1212}^{\sigma} \tau
\end{array}\right]  \tag{E6.1.5}\\
& {\left[\begin{array}{cccccc}
y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\
0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\
x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12}
\end{array}\right] }
\end{align*}
$$

Neo-Hookean Material. For a Neo-Hookean material (see Section 5.??),

$$
\begin{equation*}
C_{i j k l}^{\sigma \mathcal{T}}=\lambda \delta_{i j} \delta_{k l}+\mu(J)\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right) \tag{E6.1.6}
\end{equation*}
$$

where

$$
\begin{equation*}
J=\operatorname{det} \mathbf{F}, \quad \mu(J)=\mu_{0}-\lambda \log J, \tag{E6.1.7}
\end{equation*}
$$

and Eq. (6.4.88) is written in Voigt notation as

$$
\left[C_{a b}^{t a n}\right]=\left[\begin{array}{ccc}
\lambda+2 \mu & \lambda & 0  \tag{E6.1.8}\\
\lambda & \lambda+2 \mu & 0 \\
0 & 0 & \mu
\end{array}\right]
$$

Thus, for a Neo-Hookean material, the material tangent stiffness matrix has the same form as the stiffness matrix for small strain linear elasticity except for the dependence of the moduli on the deformation (through Eq. (6.4.90)) and the geometry factor $A_{0} \Delta A$.

Rate-Independent Elastoplasticity. For a rate-independent elastic-plastic model in terms of the Kirchoff stress, with associated plastic flow and a von-Mises yield condition, the tangent modulus is given by Eq. (5.??)

$$
\begin{equation*}
C_{i j k l}^{t a n}=C_{i j k l}^{e p}-\frac{1}{2}\left(\delta_{i l} \tau_{j k}+\tau_{i k} \delta_{j l}+\delta_{i k} \tau_{j l}+\tau_{i l} \delta_{j k}\right) \tag{E6.1.9}
\end{equation*}
$$

The elastoplastic tangent modulus is given by

$$
\begin{equation*}
C_{i j k l}^{e p}=C_{i j k l}-\frac{C_{i j m n} p_{m n} C_{k l r s} p_{r s}}{h+p_{m n} C_{m n r s} p_{r s}} \tag{E6.1.10}
\end{equation*}
$$

where $h$ is the plastic modulus, $p_{i j}=\frac{3 \tau_{i j}^{\prime}}{2 \bar{\sigma}}$ is the plastic flow direction, $\tau_{i j}^{\prime}$ is the deviatoric part of the Kirchoff stress and $\bar{\sigma}$ is the effective stress defined by (??). Assuming constant isotropic elastic moduli, Eq. (6.4.92) is written as

$$
\begin{equation*}
C_{i j k l}^{e p}=\lambda \delta_{i j} \delta_{k l}+\mu\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right)-\frac{4 \mu^{2}}{h+3 \mu} p_{i j} p_{k l} \tag{E6.1.11}
\end{equation*}
$$

Using Voigt notation and letting $\gamma=\frac{2 \mu}{(h+3 \mu)}, p_{1}=p_{11}, p_{2}=p_{22}, p_{3}=p_{12}$ and $\tau_{1}=\tau_{11}, \tau_{2}=\tau_{22}, \tau_{3}=\tau_{12}$, the tangent stiffness matrix is obtained as

$$
\begin{align*}
{\left[C_{a b}^{t a n}\right] } & {\left[\begin{array}{ccc}
\lambda+2 \mu & \lambda & 0 \\
0 & \lambda & \lambda+2 \mu
\end{array}\right] } \\
& -2 \mu \gamma\left[\begin{array}{ccc}
p_{1}^{2} & p_{1} p_{2} & p_{1} p_{3} \\
p_{2} p_{1} & p_{2}^{2} & p_{2} p_{3} \\
p_{3} p_{1} & p_{3} p_{2} & p_{3}^{2}
\end{array}\right]-\frac{1}{2}\left[\begin{array}{ccc}
4 \tau_{1} & 0 & 2 \tau_{3} \\
0 & 4 \tau_{2} & 2 \tau_{3} \\
2 \tau_{3} & 2 \tau_{3} & \tau_{1}+\tau_{2}
\end{array}\right] \tag{E6.1.12}
\end{align*}
$$

Geometric Stiffness Matrix. The geometric stiffness matrix is given by Eq. (6.3.55), i.e.,

$$
\begin{equation*}
\mathbf{K}_{I J}^{g e o}=\mathbf{I}_{2 \times 2} \int_{A_{0}} \mathcal{B}_{I}^{T} \sigma \mathcal{B}_{J} d A=\mathbf{I}_{2 \times 2} H_{I J} \tag{E6.1.13}
\end{equation*}
$$

From Eq. (E4.1.18)

$$
\mathcal{B}=\frac{1}{2 A}\left[\begin{array}{lll}
y_{23} & y_{31} & y_{12}  \tag{E6.1.15}\\
x_{32} & x_{13} & x_{21}
\end{array}\right]
$$

Substituting Eq. (E6.1.15) into Eq. (E6.1.16) gives

$$
\mathbf{H}=\frac{1}{2 A}\left[\begin{array}{cc}
y_{23} & x_{32}  \tag{E6.1.18}\\
y_{31} & x_{13} \\
y_{12} & x_{21}
\end{array}\right]\left[\begin{array}{ll}
\sigma_{x x} & \sigma_{x y} \\
\sigma_{x y} & \sigma_{y y}
\end{array}\right] \frac{1}{2 A}\left[\begin{array}{lll}
y_{23} & y_{31} & y_{12} \\
x_{32} & x_{13} & x_{21}
\end{array}\right]
$$

Assuming the integrand to be constant, the geometric stiffness matrix is obtained by multiplying the integrand in Eq. (E6.1.13) by $A$ to give

$$
\begin{align*}
\mathbf{K}_{I J}^{\text {geo }} & =\frac{1}{4 A} H_{I J} \mathbf{I}_{2 \times 2} \\
\mathbf{K}^{\text {geo }} & =\frac{1}{4 A}\left[\begin{array}{cccccc}
H_{11} & 0 & H_{12} & 0 & H_{13} & 0 \\
0 & H_{11} & 0 & H_{12} & 0 & H_{13} \\
H_{21} & 0 & H_{22} & 0 & H_{23} & 0 \\
0 & H_{21} & 0 & H_{22} & 0 & H_{23} \\
H_{31} & 0 & H_{32} & 0 & H_{33} & 0 \\
0 & H_{31} & 0 & H_{32} & 0 & H_{33}
\end{array}\right] \tag{E6.1.19}
\end{align*}
$$

The geometric stiffness matrix is independent of material response and as can be seen from Eqs. (E6.1.18-E6.1.19) depends only on the current stress rate and the geometry of the element. The load stiffness matrix is given by the same equation as described for the rod.

## Example 6.2. Two-Node Rod Element.

We now consider the two-node rod element under a state of uniaxial stress. The rod is assumed to lie along the $\hat{x}$-axis. The only non-zero Cauchy stress component is $\hat{\sigma}_{11} \equiv \hat{\sigma}_{x}$. The tangent stiffness and the external load matrices are derived in the updated Lagrangian form, i.e. in the current configuration. We first reconsider the constitutive relation for the special case of uniaxial stress. The superscript hats are dropped in the following for convenience.

The Truesdell rate of the Cauchy stress is assumed to be given by Eq. (6.3.??)

$$
\begin{equation*}
\sigma_{i j}^{\nabla \mathcal{T}}=C_{i j k \ell}^{\sigma \jmath} D_{k \ell} \tag{E6.2.1}
\end{equation*}
$$

For the case of uniaxial stress, the only non-zero components of the rate of deformation tensor are $D_{11}, D_{22}$, and $D_{33}$.

The uniaxial stress rate is therefore given by

$$
\begin{equation*}
\sigma_{11}^{\nabla \mathfrak{J}}=C_{1111}^{\sigma \mathcal{J}} D_{11}+C_{1122}^{\sigma \mathcal{J}} D_{22}+C_{1133}^{\sigma \mathfrak{y}} D_{33} \tag{E6.2.2}
\end{equation*}
$$

The traction-free condition on the surface of the rod can be stated as

$$
\begin{align*}
& \sigma_{22}^{\nabla_{c}}=C_{2211}^{\sigma \jmath} D_{11}+C_{2222}^{\sigma \jmath} D_{22}+C_{2233}^{\sigma \jmath} D_{33}=0  \tag{E6.2.3}\\
& \sigma_{33}^{\nabla}=C_{3311}^{\sigma \jmath} D_{11}+C_{3333}^{\sigma \jmath} D_{22}+C_{3333}^{\sigma \jmath} D_{33}=0
\end{align*}
$$

If the rod is initially transversely isotropic (with the axis of symmetry coincident with the $x_{1}$-axis) the tangent moduli are related by $C_{1133}^{\sigma y}=C_{1122}^{\sigma \jmath}$ and $C_{2222}^{\sigma y}=C_{3333}^{\sigma y}$. Furthermore, uniaxial stressing in the direction of the axis of isotropy preserves the transverse isotropy and these relations hold throughout the deformation. Solving Eq. (6.2.3), with these assumptions we obtain $S$

$$
\begin{equation*}
D_{22}=D_{33}, \quad D_{22}=-\frac{C_{2211}^{\sigma 3}}{C_{2222}^{\sigma \jmath}+C_{2233}^{\sigma \jmath}} D_{11} \tag{E6.2.4}
\end{equation*}
$$

Using Eq. (6.2.4) for $D_{22}$ and $D_{23}$ in Eq. (E6.2.3) gives the uniaxial relation

$$
\begin{equation*}
\sigma_{11}^{\nabla \jmath}=E^{\sigma T} D_{11} \quad \text { or } \quad\left[C^{\sigma \tau}\right]=\left[E^{\sigma \tau}\right] \tag{E6.2.5}
\end{equation*}
$$

where $E^{t a n}$ is the tangent modulus associated with the state of uniaxial stress and is given by

$$
\begin{equation*}
E^{\sigma \mathcal{T}}=C_{1111}^{\sigma \mathcal{T}}-\frac{2 C_{211}^{\sigma \tau} C_{1122}^{\sigma \tau}}{C_{2233}^{\sigma \tau}+C_{2222}^{\sigma \tau}} \tag{E6.2.6}
\end{equation*}
$$

Material Tangent Stiffness Matrix. The tangent stiffness matrix for a rate-independent material is given by Eq. (6.4.18) in the current configuration which we write in the local coordinate system as

$$
\begin{equation*}
\hat{\mathbf{K}}^{m a t}=\int_{\Omega} \hat{\mathbf{B}}^{T} \hat{\mathbf{C}}^{\sigma \tau} \hat{\mathbf{B}} d \Omega \tag{E6.2.7}
\end{equation*}
$$

Using the $\mathbf{B}$ matrix from Eq. (E4.6.3) and $\mathbf{C}^{\boldsymbol{\sigma} \boldsymbol{y}}$ as given by Eq. (E6.2.5), we obtain

$$
\hat{K}^{\text {mat }}=\int_{0}^{1} \frac{1}{\ell}\left\{\begin{array}{c}
-1  \tag{E6.2.8}\\
0 \\
1 \\
0
\end{array}\right\}\left[E^{\sigma \tau}\right] \frac{1}{\ell}\left[\begin{array}{llll}
-1 & 0 & +1 & 0
\end{array}\right] A \ell d \xi
$$

Here, the $\mathbf{B}$ matrix has been expanded to a $4 \times 1$ matrix by adding zeros to reflect that the $\hat{x}$-component of the rate-of-deformation is independent of the transverse velocities. If we assume $E^{\sigma T}$ is constant in the element, then

$$
\hat{\mathbf{K}}^{\text {mat }}=\frac{A E^{\sigma \mathcal{T}}}{\ell}\left[\begin{array}{cccc}
+1 & 0 & -1 & 0  \tag{E6.2.9}\\
0 & 0 & 0 & 0 \\
-1 & 0 & +1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

This is identical to the linear stiffness matrix for a $\operatorname{rod}$ if $E^{\sigma \tau}$ is replaced by Young's modulus $E$. The global stiffness is given by Eq. (4.5.42):

$$
\begin{equation*}
\mathbf{K}^{m a t}=\mathbf{T}^{T} \hat{\mathbf{K}}^{m a t} \mathbf{T} \tag{E6.2.10a}
\end{equation*}
$$

where $\mathbf{T}$ is given by

$$
\mathbf{T}=\left[\begin{array}{cccc}
\cos \theta & \sin \theta & 0 & 0  \tag{E6.2.10b}\\
-\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & -\sin \theta & \cos \theta
\end{array}\right]
$$

so

$$
\mathbf{K}^{\text {mat }}=\frac{A E^{\sigma \mathcal{T}}}{\ell}\left[\begin{array}{cccc}
\cos ^{2} \theta & \cos \theta \sin \theta & -\cos ^{2} \theta & -\cos \theta \sin \theta  \tag{E6.2.11}\\
& \sin ^{2} \theta & -\cos \theta \sin \theta & -\sin ^{2} \theta \\
& & \cos ^{2} \theta & \cos \theta \sin \theta \\
\text { symmetric } & & & \sin ^{2} \theta
\end{array}\right]
$$

where the material constant $E^{\sigma T}$ relates the Truesdell rate of the Cauchy stress to the rate-of-deformation in a uniaxial state of stress.

Geometric Stiffness Matrix. The geometric stiffness is developed in a coordinate system that at time $t$ coincides with the axis of the bar but is fixed in time. Note that since the coordinate system is fixed in the orientation shown in Fig. ??, it is not a true corotational coordinate system, so the rotation corrections of an objective rate must be considered. We will use the Truesdell rate. We could also consider the $\hat{x}, \hat{y}$ coordinate system corotational and derive the geometric stiffness by accounting for the channge of the transformation matrix $\mathbf{T}$ in (E4.6.11) . Many such derivations are given in Crisfield. The result should be identical, since the same mechanical effect is represented, but the derivation is generally more difficult. The geometric stiffness matrix is given by Eq. (6.4.18):.

$$
\begin{equation*}
\hat{\mathbf{K}}_{I J}=\hat{H}_{I J} \mathbf{I} \quad \hat{\mathbf{H}}=\int_{\Omega} \hat{\mathrm{B}}^{T} \sigma \mathrm{~B}^{\hat{1}} d \Omega \tag{E6.2.12}
\end{equation*}
$$

where the geometric stiffness has been expressed in the local coordinate system for simplicity. Using the $\mathcal{B}$ matrix from Eq. (4.6.3), it follows that

$$
\mathbf{H}=\int_{\Omega} \frac{1}{\ell}\left[\begin{array}{l}
-1  \tag{E6.2.13}\\
+1
\end{array}\right]\left[\hat{\sigma}_{x}\right] \frac{1}{\ell}\left[\begin{array}{ll}
-1 & +1] d \Omega \\
\hline
\end{array}\right.
$$

Assuming that the stress is constant gives

$$
\hat{\mathbf{H}}=\frac{\hat{\boldsymbol{\sigma}}_{x} A}{\ell}\left[\begin{array}{ll}
+1 & -1  \tag{E6.2.14}\\
-1 & +1
\end{array}\right]
$$

Expanding the above, we obtain the geometric stiffness as given by Eq. (E6.2.12)

$$
\hat{\mathbf{K}}^{\text {geo }}=\frac{A \hat{\sigma}_{x}}{\ell}\left[\begin{array}{cccc}
+1 & 0 & -1 & 0  \tag{E6.2.15}\\
0 & +1 & 0 & -1 \\
-1 & 0 & +1 & 0 \\
0 & -1 & 0 & +1
\end{array}\right]
$$

Use of the transformation formula, Eq. (4.5.42), shows that the geometric stiffness is independent of the orientation of the beam.

$$
\begin{equation*}
\mathbf{K}^{g e o}=\mathbf{T}^{T} \hat{\mathbf{K}}^{\text {geo }} \mathbf{T}=\hat{\mathbf{K}}^{\text {geo }} \tag{E6.2.16}
\end{equation*}
$$

The total tangent stiffness is then given by the sum of the material and geometric stiffnesses

$$
\begin{equation*}
\mathbf{K}^{\text {int }}=\mathbf{K}^{\text {mat }}+\mathbf{K}^{\text {geo }} \tag{E6.2.17}
\end{equation*}
$$

The matrix is symmetric, which is a consequence of choosing a constitutive equation in terms of the Truesdell rate of the convected rate of the Kirchhoff stress. The matrix is positive definite as long as the initial stress is small compared to the tangent modulus. THE STIFFNESS FOR JAUMANN RATE AND THE EIGENVALUES OF K ARE LEFT AS EXERCIZES.

Load Stiffness. The load stiffness for the rod is given by Eq. (??). We write only the nonzero terms noting $N,{ }_{i \eta}=0$ and that $x,_{\eta}=y,{ }_{\eta}=0$, since the shape function is only a function of $\xi$. For simplicity, we first evaluate it in the corotational system, which gives

$$
\hat{\mathbf{K}}_{I J}=\int_{0}^{1} p N_{I} N_{J, \xi}\left[\begin{array}{cc}
0 & z_{\eta}  \tag{E6.2.18}\\
-z_{\eta} & 0
\end{array}\right] d \xi
$$

In the above, $z_{\eta}$ can be taken to the width of the element $a$. Using (???) gives

$$
\hat{\mathbf{K}}_{I J}=\int_{0}^{1} p N_{I} N_{J, \xi}\left\{\left[\begin{array}{cc}
0 & 1  \tag{E6.2.19}\\
-1 & 0
\end{array}\right] a d \xi\right.
$$

Let

$$
\begin{align*}
H_{I J} & =\int_{0}^{1} N_{I} N_{J, \xi} d \xi=\int_{0}^{1}\left[\begin{array}{c}
1-\xi \\
\xi
\end{array}\right] \frac{1}{\ell}\left[\begin{array}{ll}
-1 & +1
\end{array}\right] d \xi  \tag{E6.2.20}\\
& =\frac{1}{2 \ell}\left[\begin{array}{ll}
-1 & 1 \\
-1 & 1
\end{array}\right] \tag{E6.2.21}
\end{align*}
$$

Then

$$
\begin{equation*}
\hat{K}_{I J}^{e x t}=p \ell a H_{I J} \tag{E6.2.22}
\end{equation*}
$$

Taking the shape functions (???) and substituting into the above gives

$$
\hat{\mathbf{K}}^{\text {ext }}=\frac{p a}{2}\left[\begin{array}{cccc}
0 & -1 & 0 & 1  \tag{E6.2.23}\\
1 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 \\
1 & 0 & -1 & 0
\end{array}\right]
$$

The above matrix is also invariant with rotation, i.e.,

$$
\begin{equation*}
\mathbf{K}^{\text {ext }}=\mathbf{T}^{T} \hat{\mathbf{K}}^{\text {ext }} \mathbf{T}=\hat{\mathbf{K}}^{\text {ext }} \tag{E6.2.24}
\end{equation*}
$$

for forces and velocities expressed in any other Cartesian coordinate system.
Material Tangent Stiffness Matrix in Total Lagrangian Form. The material tangent stiffness matrix for a rate-independent material is given by Eq. (6.4.18) in the reference configuration

$$
\begin{equation*}
\mathbf{K}^{m a t}=\int_{\Omega_{0}} \mathbf{B}^{T} \mathbf{C}^{S E} \mathbf{B} d \Omega_{0} \tag{E6.2.25}
\end{equation*}
$$

Using the $\mathbf{B}$ matrix from Eq. (E4.???) and $\mathbf{C}^{S E}$ as given by Eq. (E6.2.5), we obtain

$$
K^{m a t}=\int_{0}^{1} \frac{1}{\ell_{0}}\left\{\begin{array}{c}
-\cos \theta  \tag{E6.2.26}\\
-\sin \theta \\
\cos \theta \\
\sin \theta
\end{array}\right\}\left[E^{S E}\right] \frac{1}{\ell_{0}}\left[\begin{array}{llll}
-\cos \theta & -\sin \theta & \cos \theta & \sin \theta
\end{array}\right] A_{0} \ell_{0} d \xi
$$

If we assume $E^{S E}$ is constant in the element, then

$$
\mathbf{K}^{m a t}=\frac{A_{0} E^{S E}}{\ell_{0}}\left[\begin{array}{cccc}
\cos ^{2} \theta & \cos \theta \sin \theta & -\cos ^{2} \theta & -\cos \theta \sin \theta  \tag{E6.2.27}\\
& \sin ^{2} \theta & -\cos \theta \sin \theta & -\sin ^{2} \theta \\
& & \cos ^{2} \theta & \cos \theta \sin \theta \\
\text { symmetric } & & & \sin ^{2} \theta
\end{array}\right]
$$

where the material constant $E^{S E}$ relates the Truesdell rate of the Cauchy stress to the rate-of-deformation in a uniaxial state of stress. It can easily be shown via (???) that the above is identical to (E6.2.12).

Geometric Stiffness Matrix in Total Lagrangian Form. The geometric stiffness is developed from (6.4.15):

$$
\begin{equation*}
\mathbf{K}_{I J}=H_{I J} \mathbf{I} \quad \mathbf{H}=\int_{\Omega_{0}} \mathrm{~B}_{0}^{T} \mathbf{S} \mathrm{~B}_{0} d \Omega_{0} \tag{E6.2.28}
\end{equation*}
$$

where the $B_{0}$ matrix is given in (4.6.3), so

$$
\mathbf{H}=\int_{\Omega_{0}} \frac{1}{\ell_{0}}\left[\begin{array}{l}
-1  \tag{E6.2.29}\\
+1
\end{array}\right]\left[\begin{array}{l}
\left.S_{11}\right]
\end{array} \frac{1}{\ell_{0}}\left[\begin{array}{ll}
-1 & +1
\end{array}\right] d \Omega_{0}\right.
$$

Assuming that the stress is constant gives

$$
\hat{\mathbf{H}}=\frac{S_{11} A_{0}}{\ell_{0}}\left[\begin{array}{ll}
+1 & -1  \tag{E6.2.30}\\
-1 & +1
\end{array}\right]
$$

Expanding the above, we obtain the geometric stiffness

$$
\mathbf{K}^{\text {geo }}=\frac{A_{0} S_{11}}{\ell_{0}}\left[\begin{array}{cccc}
+1 & 0 & -1 & 0  \tag{E6.2.31}\\
0 & +1 & 0 & -1 \\
-1 & 0 & +1 & 0 \\
0 & -1 & 0 & +1
\end{array}\right]
$$

The total tangent stiffness is then given by the sum of the material and geometric stiffnesses, (E6.2.17).
6.3.7. Constraints. Three types of methods are frequently used for treating the constraint Eq. (6.3.10). They are:

1. penalty methods
2. Lagrangian multiple methods
3. augmented Lagrangian methods

These methods originate in constrained optimization theory. As will be seen, they can readily be adapted to the solution of the nonlinear algebraic equations that correpond to the momentum or equilibrium equations, Eq. (6.3.10). To motivate these methods, we begin with a description of how they are applied to the nonlinear minimization problem, Eq. (6.3.27) (note that while Eq. (6.3.27)
[...] the stationary prints are determined in the problem is often called a minimization problem because often only the stable equilibrium solutions are of interest).

The problem then is to solve:

$$
\mathbf{r}(\mathbf{d})=0 \quad \text { subject to } g_{I}(\mathbf{d})=0, I=1 \text { to } n_{c}
$$

where $\mathbf{r} \in R^{n}, \mathbf{d} \in R^{n}$. The following notation is used

$$
\begin{aligned}
& W_{,_{a}}=\frac{\partial W}{\partial d_{a}} \quad G_{I a}=\frac{\partial g_{I}}{\partial d_{a}}=g_{I, a} \quad G_{I a}=\frac{\partial g_{I}}{\partial d_{a}}=g_{I, a} \\
& r_{a, b}=A_{a b} \quad \text { or } \quad \frac{\partial \mathbf{r}}{\partial \mathbf{d}}=\mathbf{A}=\mathbf{M} \ddot{\mathbf{d}}+\mathbf{f}^{\text {int }}-\mathbf{f}^{e x t}
\end{aligned}
$$

We will use the conventions $G_{I}=\left[G_{I 1}, G_{I 2}, \ldots, G_{I n_{c}}\right]$ and $\mathbf{H}_{I} \in R^{n_{c}} \times R^{n_{c}}$, as before. Recall that

$$
\begin{equation*}
W,_{a}=r_{a}=f_{a}^{\mathrm{int}}-f_{a}^{e x t} \tag{6.3.40d}
\end{equation*}
$$

in a conservation problem and that $W,{ }_{a b}$, the Jacobian

$$
\begin{equation*}
W,_{a b}=A_{a b} \tag{6.3.40e}
\end{equation*}
$$

matrix of the system.
We will also examine the less general problem of finding stationary the points of

$$
\begin{equation*}
W(\mathbf{d})=0 \text { subject to } g_{I}(\mathbf{d})=0 \tag{6.3.41}
\end{equation*}
$$

Lagrange Multiplier Method. In the Lagrange method, the constraints are appended to the objective function with the Lagrangian multipliers. The minimization Eq. (6.3.41) becomes equivalent to finding the stationary points of

$$
\begin{equation*}
W+\lambda_{I} g_{I} \equiv W+\lambda^{T} \mathbf{g} \tag{6.3.42}
\end{equation*}
$$

Note that at a minimum of $W$, the augmented function given above has a saddle point.

The stationary points are found by setting to zero the derivatives of the above with respect to $d_{a}$ and $\lambda_{I}$ :

$$
\begin{equation*}
W_{,_{a}}+\lambda_{I} g_{I, a} \equiv r_{a}+\lambda_{I} g_{I, a}=0 \tag{6.3.43a}
\end{equation*}
$$

$$
\begin{equation*}
g_{I}=0 \tag{6.3.43b}
\end{equation*}
$$

The above is the system of equations for the constrained problem. The constraint introduces extra forces $\lambda_{I} g_{I, a}$, which are linear combinations of the Lagrange multipliers.

The linear model of (6.3.43) is the first two terms of a Taylor expansion of (6.3.43), giving

$$
\begin{align*}
& r_{a}+\lambda_{I} g_{I, a}+r_{a, b} \Delta d_{b}+\lambda_{I} g_{I, a}+\lambda_{I} g_{I, a b} \Delta d_{b}=0  \tag{6.3.44a}\\
& g_{I}+g_{I, a} \Delta d_{a}=0 \tag{6.3.44b}
\end{align*}
$$

Using matrix notation we can write the above as

$$
\left[\begin{array}{cc}
\mathbf{A}+\lambda_{I} \mathbf{H}_{I} & \mathbf{G}^{T}  \tag{6.3.45}\\
\mathbf{G} & \mathbf{0}
\end{array}\right]\left\{\begin{array}{l}
\Delta \mathbf{d} \\
\Delta \lambda
\end{array}\right\}=\left\{\begin{array}{cc}
-\mathbf{r}-\lambda^{T} \mathbf{G} & \\
& -\mathbf{g}
\end{array}\right\}
$$

So the linear model has $n_{c}$ additional equations due to the constraint. Even when the $\mathbf{A}$ is positive definite, the augmented system of equations will not be positive definite because of the zeroes on the diagonal in the lower right hand corner of the matrix. For a linear statics problem with a linear constraints $\mathbf{G d}=\mathbf{g}$, the above becomes

$$
\left[\begin{array}{cc}
\mathbf{K} & \mathbf{G}^{T}  \tag{6.3.46}\\
\mathbf{G} & \mathbf{0}
\end{array}\right]\left\{\begin{array}{l}
\mathbf{d} \\
\lambda
\end{array}\right\}=\left\{\begin{array}{c}
\mathbf{e}^{e x t} \\
\mathbf{g}
\end{array}\right\}
$$

since

1. $\mathbf{A}=\mathbf{K}$ for linear statics;
2. $\mathbf{H}_{I}=0$ for linear constraints, see Eq. (6.3.40c);
3. the starting value is zero and $\Delta \mathbf{d}=\mathbf{d}, \Delta \lambda=\lambda$, and the constraint is $\mathbf{G d}=\mathbf{0}$.

For the general problem with nonconservative materials, dynamics, etc., the Lagrangian multiplier method is formulated as follows. The stationary condition Eq. (6.3.43) can be written

$$
\begin{equation*}
\delta W+\delta\left(\lambda_{I} g_{I}\right)=0 \tag{6.3.47}
\end{equation*}
$$

From Eq. (B4.6.1) and Eq. (6.3.1)

$$
\begin{align*}
\delta W & =\delta W^{\text {ext }}-\delta W^{\text {int }}+\delta W^{\text {inert }} \\
& =\delta \mathbf{d}^{T}\left(-\mathbf{f}^{\text {ext }}+\mathbf{f}^{\text {int }}+s_{D} \mathbf{M} \ddot{\mathbf{d}}\right)  \tag{6.3.48}\\
& =\delta \mathbf{d}^{T} \mathbf{r}=\delta d_{a} r_{a}
\end{align*}
$$

Substituting Eq. (6.3.48) into Eq. (6.3.47) and writing out the differentials on the second term gives

$$
\begin{equation*}
\delta \mathbf{d}_{a} r_{a}+\delta \lambda_{I}^{T} g_{I}+\lambda_{I}^{T} g_{I, a} \delta d_{a}=0 \quad \forall \delta d_{a} \delta \lambda_{I} \tag{6.3.49}
\end{equation*}
$$

Using the arbitrariness of the differentials in the above implies Eq. (4.3.44-45). Thus the same structure is obtained for a nonconservative dynamic problem. The linearization procedure leads to the same equations, Eq. (6.3.45). While the development has been given for the virtual work $\delta W$, it applies identically to virtual power.

Penalty Method. Again, we first consider conservative problems where the solution is determined by minimization. In the penalty method, the constraint is enforced by adding the square of the constraints to the poential, so we minimize as modified potential

$$
\begin{equation*}
\bar{W}(\mathbf{d})=W(\mathbf{d})+\frac{1}{2} \beta g_{I}(\mathbf{d}) g_{I}(\mathbf{d}) \tag{6.3.50}
\end{equation*}
$$

where $\beta$ is a penalty parameter. The penalty parameter is generally orders of magnitude greater than other parameters of the problem. The idea is that if $\beta$ is large enough, the minimum of $\bar{W}(\mathbf{d})$ cannot be attained without satisfying the constraints.

The stationary (or minimum) conditions give

$$
\begin{equation*}
\bar{W},_{a}=W,_{a}+\beta g_{I} g_{I, a}=0 \quad \text { or } \quad \mathbf{r}+\beta \mathbf{g}^{T} \mathbf{G}=0 \tag{6.3.51}
\end{equation*}
$$

The linear model is

$$
\begin{equation*}
\left(r_{a, b}+\beta g_{I, b} g_{I, a}+g_{I} g_{I, a b}\right) d_{b}=1\left(r_{a}+\beta g_{I} g_{I a}\right) \tag{6.3.52}
\end{equation*}
$$

or in matrix form

$$
\begin{equation*}
\bar{A} \Delta \mathbf{d}=\left(\mathbf{A}+\beta \mathbf{G}^{T} \mathbf{G}+g_{I} \mathbf{H}_{I}\right) \Delta \mathbf{d}_{b}=-\mathbf{r}+\beta \mathbf{g}^{T} \mathbf{G} \tag{6.3.53}
\end{equation*}
$$

The size of this system is not increased over the unconstrained system. For linear constraints, if $\mathbf{A}>0, \bar{A}>0$, i.e. the augmented system if positive definite if the original Jacobian matrix is positive definite. The major drawback of penalty methods is that they impair the conditioning of the equations.

The discrete equations for nonconservative systems are obtained by the same procedure as for the Lagrange multipliers. Write the stationary conditions in differential form

$$
\begin{equation*}
0=\delta \bar{W}=\delta W+\frac{1}{2} \beta \delta\left(g_{I} g_{I}\right) \tag{6.3.54}
\end{equation*}
$$

Now apply Eq. (6.3.48) to replace $\delta W$ in the above. The discrete equations and linear model are then given by Eqs. (6.3.51) and (6.3.52), respectively.

### 6.5. STABILITY: CONTINUATION AND ARCLENGTH METHODS

Stability. In nonlinear problems, stability of solutions is of considerable interest. There are many definitions of stability: stability is a concept that depends on the beholder and his objectives. However, some general definitions are widely accepted. We will here describe a theory of stability that originates from Liapunov(??) and is widely used throughout mathematical analysis, see Saybol(??) for a very lucid account of its computtional application to a variety of problems. We will focus on its application to finite element methods.

We will first give a definition of stability and explore its implications. Consider a process that is governed by an evolution equation such as the equation of motion or the heat conduction equations. Let the solution for the initial conditions $\mathbf{d}_{A}(0)=\mathbf{d}_{A}^{0}$ be denoted by $\mathbf{d}_{A}(t)$. Now consider additional solutions for initial conditions $\mathbf{d}_{B}(0)=\mathbf{d}_{B}^{0}$, where $\mathbf{d}_{B}^{0}$ are small perturbation of $\mathbf{d}_{A}^{0}$. This means that $\mathbf{d}_{B}^{0}$ is close to $\mathbf{d}_{A}^{0}$ in some norm, i.e.

$$
\begin{equation*}
\left\|\mathbf{d}_{A}^{0}-\mathbf{d}_{B}^{0}\right\| \leq \varepsilon \tag{6.5.1}
\end{equation*}
$$

A solution is stable when for all initial conditions that satisfy (6.5.1), the solution satisfies

$$
\begin{equation*}
\left\|\mathbf{d}_{A}(t)-\mathbf{d}_{B}(t)\right\| \leq C \varepsilon \quad \forall t>0 \tag{6.5.2}
\end{equation*}
$$

To explain this definition, we specify the norm to be the $\ell_{2}$ norm. Note that all initial conditions which satisfy (6.5.1) lie in a neighborhood of $\mathbf{d}_{A}^{0}$. It is often said that the initial conditions lie in a ball around $\mathbf{d}_{A}^{0}$. The definition then states that for all initial conditions which lie in the ball around $\mathbf{d}_{A}^{0}$, the solutions $\mathbf{d}_{B}(t)$ must lie in a ball around the solution $\mathbf{d}_{A}(t)$ for all time. This definition is illustrated for a system with two dependent variables in Fig. 6.7. The right-hand
side shows the behavior of a stable system. Here we have only shown two solutions resulting from perturbations of the initial data, since it is impossible to show an infinite number of solutions. The leftr hand side shows an unstable system. It suffices for a single solution starting in the ball about $\mathbf{d}_{A}^{0}$ to diverge to indicate an unstable solution.

The applicability of this definition to processes we intuitively consider stable and unstable can be seen by the following examples. Consider a beam a beam loaded axially by a vertical load as shown in Fig. 6.8. We first consider the numerical response when the beam is perfectly straight. The lateral response in this case is the path is denoted by $d_{A}(t)$, and as can be seen, the lateral displacement is zero even though the load eventually exceeds the buckling load. If you don't believe this, try it. The beam will usually not buckle in an incremental solution or a dynamic solution with explicit or implicit integration. Only if roundoff error introduces a "numerical imperfection" will the perfectly straight beam buckle. We then plot the lateral displacement of the beam as we perturb the location of node 2 , which can be considered an initial condition on the displacement of that node. The resulting paths are also shown in Fig. 6.8. It can be seen that when the load is below the buckling load, the paths for different initial conditions remain in a ball about the $d_{A}(t)$. However, when the load exceeds the buckling load, the solutions for different initial conditions in the location of point A diverge drastically. Therefore any process in which the load exceeds the buckling load is unstable. Note that the direction of the divergence depends on the sign of the initial imperfection.

Another example is the flow of a liquid in a pipe. When the flow velocity is below a critical Reynold's number, the flow is stable. A perturbation of the state leads to small changes in the evolution of the system. On the other hand, when the flow is above the critical Reynold's number, a small perturbation leads to large changes because the flow changes from laminar to turbulent.

Stability is usually ascribed to a state, rather than a process. The definition is then identical: a state is stable if a small perturbation of that state results in a small differences for all time. When perturbations lead to large differences in the subsequent states of the system, the state is unstable. This concept fits within the framework of the definition of stability given by Eq. (6.5.1) with the state considered as the initial condition.

A common example of stable and unstable states often given in introductory dynamics texts is shown in Fig. 6.9. It is clear that state $A$ is stable, since small perturbations of the positionn of the ball will not significantly change the evolution of the system. State $B$ is unstable, small perturbations will lead to large changes: the ball can roll either to the right or to the left. State $C$ is often called neutral stability in introductory texts. According to the definition of Eq. (6.5.1), state $C$ is an unstable state, since small changes in the velocity will lead to large changes in the position as large times. Thus the definition of stability given in introductory texts does not completely conform to the one given here.

Stability of Equilibrium Solutions. To obtain a good understanding oof the behavior of a system, its equilibrium paths, or branches, and their stability
must be determined. It is often argued among structural mechanicians that the difficulties associated with unstable behavior can be circumvented by simply obtaining a dynamic solution. When a structure is loaded above its limit point or a bifurcation point in a dynamic simulation, the structure passes dynamically to the vicinity of the next stable branch and the instability is not apparent except for the onset of a different mode of deformation, such as the lateral deformation in a beam. However, to understand the behavior of a structure or process thoroughly, its static equilibrium behavior should be carefully examined. Many vagaries of structural behavior may be hidden by dynamic simulations. For example, when the fundamental path bifurcates with an asymmetric branch as shown in Fig. 6.10 , the structure becomes very sensitive to imperfections. The theoretical bifurcation point is not a realistic measure of the strength; an actual structure will buckle at a much lower load than the theoretical value because imperfections are unavoidable. A single numerical simulation could miss this sennsitivity completelly. This sensitivity to imperfections for cylindrical shells was analyzed by Koiter(??) and is a classical example of imperfection sensitivity.

As a first step in studying the equilibrium behavior of a system, the load and any other parameters of interest, such as the temperature or an active control, must be parametrized. Up to now we have parametrized the load by the time $t$, which is convenient in many practical problems. However, a single parameter does not always suffice in the study of equilibrium problems. We will now parametrize the load by $n_{\gamma}$ parameters $\gamma_{a}$, so the load is then given by $\gamma_{a} q_{a}$, where $q_{a}$ represent a distributed loading such as a pressure or concentrated loads. We use the convention that repeated indices are summed over the range, in this case $n_{\gamma}$. For distributed loads, the parameter $\gamma_{a}$ should not be applied directly to the external nodal forces, since the external nodal forces will depend on the nodal displacements. The discrete loads can be parametrized by $\gamma_{a} \mathbf{f}_{a}^{\text {ext }}$, where $\mathbf{f}_{a}^{\text {ext }}$ are column matrices of nodal external forces associated with a loading mode $a$.

Our intention is to trace the equilibrium behavior of the model as a function of the parameters $\gamma_{\alpha}$ The problem then is then is to find $\mathbf{d}\left(\gamma_{a}\right)$ such that

$$
\begin{equation*}
\mathbf{r}\left(\mathbf{d}\left(\gamma_{a}\right)\right)=\mathbf{0} \tag{6.5.2b}
\end{equation*}
$$

For purposes of characterizing the nonlinear system, the solutions are usually grouped into branches, which are continuous lines describing the response for one change of one parameter. Branches along which the solution is in equilibrium, i.e. satisfies $\mathrm{Eq}(6.5 .2 \mathrm{~b})$, are called equilibrium branches, regardless of whether they are stable or unstable.

Nonlinear systems exhibit three types of branching behavior:

1. turning points, usually called limit points in structural analysis, in which the slope of the branch changes sign;
2. stationary bifurcations, often called simply bifurcations, in which two equilibrium branches intersect.
3. Hopf bifurcations, in which an equilibrium branch intersects with a branch on which there is periodic motion.

The behavior of the shallow truss exhibits a limit (or turning) point, as can be seen from Fig. 6.11. Subsequent to a turning point, a branch can be either stable or stable. In this case, as shown in the analysis of the problem in Example 6.4 , the branch after the first limit point, point $a$, is unstable, while the branch after thhe second limit point, point $b$, is stable.

The beam problem shown in Fig. 6.12 is a classical example of a bifurcation. The point $b$ where the two branches intersect is the point of bifurcation. Subsequent to the bifurcation point, the continuation of the fundamental branch $a b$, becomes unstable. Point $b$, the bifurcation point, corresponds to the classical buckling load of the Euler beam. This type of branching is often called a pitchfork, (do you see the hay on the end?)

Hopf bifurcations are quite uncommon in passive structures. They are found in general nonlinear behavior and can be seen in structures under active control. In a Hopf bifurcation, stable equilibtrium solutions become impossible at the end of a branch. Instead, there are two branches with periodic solutions. An example of a Hopf bifurcation is given in Example ??.

Methods of continuation and arclength methods. The tracing of branches is called a continuation method. The tracing of equilibrium branches is often quite difficult and robust, automatic procedures for continuation are not available. In the following, we describe continuation methods base on parametrization, such as the arc length method. In the arc length method, the arc length along the equilibrium path replaces the load as the incremental parameter. It enables branches to be followed around turning points, which is critical to the succesful continuation of equilibrium branches..

We first consider continuation with the arc length method for the case of a single load parameter. In tracing the branches in a model with a single load parameter, the load parameter is usually started at zero and incremented. For each load increment, an equilibrium solution is computed, i.e. we find $\mathbf{d}^{n+1}$, a solution to

$$
\begin{equation*}
\mathbf{r}\left(\mathbf{d}^{n+1}, \gamma^{n+1}\right)=0 \text { or } \quad \mathbf{f}^{i n t}\left(\mathbf{d}^{n+1}\right)-\gamma^{n+1} \mathbf{f}^{e x t}=0 \tag{6.5.3}
\end{equation*}
$$

where $n$ is the step number and $\mathbf{f}^{e x t}$ is the load distribution chosen for tracing the
branch. We assume that the loads are prescribed discretely so that the distribution of nodal external forces does not change with the deformation of the model. The inertial term is not included in the above because continuation methods are applicable only to equilibrium problems. One of the most widely used continuation methods in structural mechanics is the arc length method. Instead of incrementing the load parameter $\gamma$ to trace the branch, a measure of the arclength is incremented. This is accomplished by adding a parametrization equation to the equilibrium equations

$$
\begin{equation*}
p(\mathbf{d}, \Delta \mathbf{d}, \gamma)=\Delta s^{2}, \quad \Delta \gamma=\gamma^{n+1}-\gamma^{n}, \quad \Delta \mathbf{d}=\mathbf{d}^{n+1}-\mathbf{d}^{n} \tag{6.5.4}
\end{equation*}
$$

The parametrization equations may be written in terms of the displacements or increments in the displacements or both. For example, in the arclength method the parametrizationn equations are written directly in terms of the displacement increments

$$
\begin{equation*}
\Delta \mathbf{d}^{T} \Delta \mathbf{d}+\Delta \gamma^{2}=\Delta s^{2} \tag{6.5.4b}
\end{equation*}
$$

Many other types of parametrization equations can be devised, and some are described at the end of this section. DEESCRIBED FISH PARAMETRIZATION LATER WHEN SCALINNG IS DESCRIBED

The total system of equations then consists of the equilibrium equations and the parametrization equation, so we have

$$
\left\{\begin{array}{c}
\mathbf{r}(\mathbf{d}, \gamma)  \tag{6.5.5}\\
p(\mathbf{d}, \Delta \mathbf{d}, \gamma)
\end{array}\right\}=\left\{\begin{array}{c}
\mathbf{0} \\
\Delta s^{2}
\end{array}\right\}
$$

The load parameter $\gamma$ is now treated as an additional unknown of the system and the arclength $s$ is now incremented instead of the parameter $\gamma$.

This procedure is most easily explained for a one degree-of-freedom problem such as the shallow truss shown in Fig. 6.13. The fundamental branch is shown in Fig. 6.13 and we assume that a solution has been obtained at point $n$. The arclength equation when viewed in the $\gamma, d_{y}$ is the circle about point $n$; in the 3space $\left(\gamma, d_{x}, d_{y}\right)$ it would be a sphere about the point. In solving the parametrized equations, (6.5.5), we seek a solution which is the intersection of the equilibrium branch with the circle about the last solution point, which gives the solution shown as point $n+1$ in Fig. 6.13. Thus, while incrementing the load parameter would be fruitless at point $n$, the problem has been restated in terms of the arclength along the branch so that a solution with a lower load can be found.

The parametrized equations for the truss with symmetry can then be posed as follows: find a solution to
$r_{1}\left(d_{1}, \gamma\right)=-f_{1}\left(d_{1}, \gamma\right)=0 \quad$ subject to $\left(\gamma(s)-\gamma^{n}\right)^{2}+\left(d_{1}(s)-d_{1}^{n}\right)^{2}=\Delta s^{2}$
Alternatively, we can write the above in terms of increments in the displacements and the load parameters as: find a solution to

$$
\begin{equation*}
f_{1}=0 \quad \text { subject to } \Delta \gamma^{2}+\left(\Delta d_{1}\right)^{2}=\Delta s^{2} \tag{6.5.7}
\end{equation*}
$$

Thus the problem with one unknown is augmented by a second equation, which leads to two nonlinear algeraic equations in two unknowns. The load need not increase in the step, and may in fact decrease. It is only necessary for the arclength parameter to increase, which is a perfectly natural way of tracing the branch.

To describe the method in a more general case, we consider the problem with $n_{\gamma}$ load parameters $\gamma_{a}$. For each load parameter, a parametrization equation must be added:

$$
\begin{equation*}
p_{a}\left(\mathbf{d}, \gamma_{b}\right)=\Delta s_{a}^{2} \quad \text { or } \quad \mathbf{p}(\mathbf{d}, \gamma)=\Delta \mathbf{s} \tag{6.5.8}
\end{equation*}
$$

where

$$
\mathbf{p}=\left[\begin{array}{lll}
p_{1} & \ldots & p_{n_{\gamma}}
\end{array}\right], \quad \Delta \mathbf{s}=\left[\begin{array}{lll}
\Delta s_{1}^{2} & \ldots & \Delta s_{n_{\gamma}}^{2}
\end{array}\right], \gamma=\left[\begin{array}{lll}
\gamma_{1} & \ldots & \gamma_{n_{\gamma}}
\end{array}\right]
$$

The resulting augmented equations for the equilibrium path are then

$$
\left\{\begin{array}{c}
\mathbf{f}(\mathbf{d}, \gamma)  \tag{6.5.9}\\
\mathbf{p}(\mathbf{d}, \gamma)
\end{array}\right\}=\left\{\begin{array}{c}
\mathbf{0} \\
\Delta \mathbf{s}
\end{array}\right\}
$$

Thus for a system with $n_{D O F}$ degrees of freedom, we obtain an augmented system of $n_{D O F}+n_{\gamma}$ equations in the same number of unknowns.

The resulting equations can be solved by the standard Newton methods we have described. The linearized equations for the Newton method are given by

$$
\begin{align*}
& {\left[\begin{array}{cc}
\partial \mathbf{f} / \partial \mathbf{d} & \partial \mathbf{f} / \partial \gamma \\
\partial p / \partial \mathbf{d} & \partial \mathbf{p} / \partial \gamma
\end{array}\right]\left\{\begin{array}{c}
\Delta \mathbf{d} \\
\Delta \gamma
\end{array}\right\}=\left\{\begin{array}{c}
\mathbf{0} \\
\Delta \mathbf{s}
\end{array}\right\}} \\
& {\left[\begin{array}{cc}
\mathbf{K}^{\text {int }}-\gamma_{a} \mathbf{K}_{a}^{e x t} & \partial \mathbf{p} / \partial \gamma \\
\partial \mathbf{p} / \partial \mathbf{d} & \partial \mathbf{p} / \partial \gamma
\end{array}\right]\left\{\begin{array}{c}
\Delta \mathbf{d} \\
\Delta \gamma
\end{array}\right\}=\left\{\begin{array}{c}
\mathbf{0} \\
\Delta \mathbf{s}
\end{array}\right\}} \tag{6.5.10}
\end{align*}
$$

where the Jacobians of the nodal forces have been expressed in terms of the internal tangent stiffness and the load stiffness on the LHS. A subscript has been added to the load stiffness because the Jacobian for each group of external loads must be considered separately. At times the internal tangent stiffness must also be subdivided into terms associated with different parameters, as when the temperature changes and causes buckling.

The parametrization equations need not be arclength equations: any parametrization which leads to a regular set of Newton equations is a candidate.

A major difficulty in all branch continuation techniques is setting the increment size and in the scaling of the parametrization equations. If the increment size is too small, considerable effort is wasted in determining unnecessary equilibrium points. On the other hand, if the increment size is too large, the convergence of the Newton procedure can fail or too many iterations are needed. The selection of an appropriate stepsize can be aided by an estimate of the location of the next turning point or bifurcation point. The step size can then be set so that a reasonable fraction of that distance is covered in the next increment. It is stressed however that bifurcation paths can appear out of nowhere, so if a good knowledge of the branches is essential, branch continuations should be repeated with different stepsizes.

Scaling of Arclength Equation. The arclength equation, when posed in terms of force parameters and displacement increments is often poorly scaled. We summarize in the following some of the scaling methods which have been proposed that appear to be effective in structural mechanics problems.

The simplest method is to introduce a scaling factor between the increments in load and the increments in displacements. The parametrization then is

$$
\begin{equation*}
p(\mathbf{d}, \gamma)=\Delta \gamma^{2}\left(\Delta \mathbf{f}_{0}^{T} \Delta \mathbf{f}_{0}\right)+\alpha \Delta \mathbf{d}^{T} \Delta \mathbf{d}=\Delta s^{2} \tag{6.5.11}
\end{equation*}
$$

where $\alpha$ is a scaling factor. A candidate for a scaling factor is the square of the average diagonal of the initial stiffness matrix.

Bifurcations. We consider first equilibrium bifurcations, i.e. we ignore Hopf bifurcations. The bifurcation then consists of the intersection of two equilibrium branches. If we are tracing a given equilibrium branch, such as AB in Fig. ?, then it is very easy to miss the intersecting branch and end up on an unstable branch. The objective of this Section is to describe some methods for detecting the crossing of a bifurcation point and anticipating when a bifurcation point will come up along thhe branch.

The classical method for detecting bifurcations in structural mechanics is an eigenvalue analysis. In an eigenvalue approach, we exploit the fact that the linearized equations for the increment, Eqs. () are singular at the bifurcation point.

Linear Stability. In Example 1, we have employed a technique which is frequently used to examine the stability of an equilibrium solution: a dynamic solution to a perturbation of the equilibrium solution. The dynamic equations are linear because the perturbations are small, so this is called a linearized model. If the dynamic solution grows, then it is said that the equilibrium solution is linear unstable. Otherwise, it is linear stable. In the following, we develop a general procedure for examining the linear stability of an equilibrium solution by examining the characteristics of the Jacobian matrix.

We consider an equilibrium point $\mathbf{d}^{e q}$ associated with a parametrized load, $\lambda \mathbf{f}^{e x t}$ of a rate-independent system. A Taylor series expansion of the residual about the equilibrium solution gives

$$
\begin{equation*}
\mathbf{f}\left(\mathbf{d}^{e q}+\overline{\mathbf{d}}\right)=\mathbf{f}\left(\mathbf{d}^{e q}\right)+\frac{\partial \mathbf{f}}{\partial \mathbf{d}}\left(\mathbf{d}^{e q}\right) \overline{\mathbf{d}}+\text { higher order terms } \tag{6.5.12}
\end{equation*}
$$

The first term on the RHS vanishes because $\mathbf{d}^{e q}$ is an equilibrium solution. From Eq. () we can see that the second term can be linearized as follows:

$$
\begin{equation*}
\frac{\partial \mathbf{f}}{\partial \mathbf{d}}\left(\mathbf{d}^{e q}\right)=\mathbf{K}^{e x t}\left(\mathbf{d}^{e q}\right)-\mathbf{K}^{i n t}\left(\mathbf{d}^{e q}\right)=-\overline{\mathbf{A}}\left(\mathbf{d}^{e q}\right) \tag{6.5.13}
\end{equation*}
$$

We now add the inertial forces to the system. Since the mass matrix does not change with displacements, we can then write the equations of motion for a small perturbations about the equilibrium point as

$$
\begin{equation*}
\mathbf{M} \frac{d^{2} \overline{\mathbf{d}}}{d t^{2}}+\overline{\mathbf{A}} \overline{\mathbf{d}}=0 \tag{6.5.14}
\end{equation*}
$$

Note, that in contrast to Section ??, we do not include the mass matrix in the Jacobian matrix $\overline{\mathbf{A}}$. The above is a set of linear ordinary differential equations in $\overline{\mathbf{d}}$. Since the solutions to such linear ordinary differential equations are exponential, we take solutions of the form

$$
\begin{equation*}
\overline{\mathbf{d}}=\mathbf{y} e^{\mu t} \quad \bar{d}_{i}=y_{a} e^{\mu t} \tag{6.5.15}
\end{equation*}
$$

Substituting the above into Eq. (6.5.14) gives

$$
\begin{equation*}
\left(\mathbf{A}+\mu^{2} \mathbf{M}\right) \mathbf{y} e^{\mu t}=0 \tag{6.5.16}
\end{equation*}
$$

The characteristic values $\mu_{i}$ of this system can be obtained from the eigenvalue problem

$$
\begin{equation*}
\mathbf{A} \mathbf{y}_{i}=-\lambda_{i} \mathbf{M} \mathbf{y}_{i}, \quad \lambda_{i}=\mu_{i}^{2} \tag{6.5.17}
\end{equation*}
$$

where $\lambda_{i}, i=1$ to $n$ are the $n$ eigenvalues and $y_{i}$ are the $n$ eigenvectors. The linear stability of the system then revolves around the character and magnitudes of the eigenvalues $\mu_{i}$. The eigenvalues will generally be complex. If the real part of the eigenvalue is positive the solution will grow, i.e. if
if for any $i, \operatorname{Real}\left(\mu_{i}\right)>0$, the equilibrium point is linearly unstable(6.5.18)

Here $\bar{\mu}_{i}$ is the complex conjugate of $\mu_{i}$. On the other hand, if the real parts of all eigenvalues are negative, then the linearized solutions about the equilibrium point do not grow and we can say that

$$
\begin{equation*}
\text { if for all } i \text {, } \operatorname{Real}\left(\mu_{i}\right) \leq 0 \text {, equilibrium point is linearly stable } \tag{6.5.19}
\end{equation*}
$$

When the linearized equations are symmetric, then the eigenvalues must be real. We can then see that if the matrix $\mathbf{A}$ is positive definite, then the eigenvalues must be negative, and consequently the parameters $\mu_{i}$ are strictly complex. Therefore, when $\mu$ are complexthe perturbated solutions are harmonic and of the same magnitude as the perturbation and the equilibrium points is linear stable.

This result has important ramifications for many structural systems. If the system has a potential, i.e. if the system is conservative, then the matrix $\mathbf{A}$ is symmetric and corresponds to the Hessian of the potential energy, i.e., $A_{a b}=\partial^{2} W / \partial d_{d} \partial d_{b}$ by Eq. ???. Recall that an equilibrium solution is a stationary point of the potential. Since $\mathbf{A}$ is the matrix of second derivatives, the positive definiteness of $\mathbf{A}$ implies that the stationary point is a local minimum. Thus an equilibrium point is linear stable if and only if the potential at the equilibrium point is a local minimum, which implies that the Jacobian and Hessian matrices are positive definite. In other words, if

$$
\Delta d_{a} \frac{\partial^{2} W\left(\mathbf{d}^{e q}\right)}{\partial d_{a} \partial d_{b}} \Delta d_{b}=A_{a b}\left(\mathbf{d}^{e q}\right) \Delta d_{a} \Delta d_{b}=\Delta \mathbf{d}^{T} \mathbf{A} \Delta \mathbf{d}>0 \quad \forall \Delta \mathbf{d} \text { (6.5.20) }
$$

then the equilibrium point $\mathbf{d}^{e q}$ must be linear stable. On the other hand, if there exists a $\Delta \mathbf{d}$ for which the above inequality is violated, then the stationary point must be a saddle point, and the equilibrium solution is not linear stable.

To summarize, an equilibrium solution for a conservative system is linear stable if it corresponds to a local minimum of the potential energy, which requires the positive definiteness of the Hessian or Jacobian matrices (they are the same in that case). If the equilibrium solution is a saddle point, then the equilibrium solution is unstable.

For nonconservative systems, an equilibrium solution is also linear stable if the Jacobian matrix is symmetric and positive definite. If the Jacobian is not positive definite, it is not linear stable. Any system is linear stable if all real parts of the eigenvalues of the system (6.5.17) are negative.

The information provided by a linear stability analysis is not conclusive from an engineering viewpoint. Since linear stability analysis assumes the linearity of the response in the vicinity of the equilibrium point, perturbations must be small enough so that the response can be predicted by a linear model. Linear stability of an equilibrium point does not preclude the possibility that a physically realistic perturbation will grow. If the system is highly nonlinear in the
neighborhood of the equilibrium point, moderate perturbations of the system may lead to unstable growth. A linear stability analysis only reveals how a system with properties obtained by a linearization of the system about the equilibrium point behaves. Nevertheless, it gives information which is useful in engineering and scientific analysis of systems.

Estimates of Bifurcation Points. It is often desirable to determine the location of bifurcation points as the equilibrium path is generated. Both bifurcation points which may have been passed or which are upcoming are of interest. Whether a bifuraction point has been passed can be determined by checking when the determinant of the Jacobian changes sign. A change of sign in the determinant of the Jacobian is an indication of the change of sign of an eigenvalue. The determinant of the Jacobian vanishes at a critical point and will often change sign when the critical point is passed. It would appear at first that the sign of the Jacobiann determinant would always change when passing a bifurcation but things are not that simple: sometimes, two eigenvalues change sign at a bifurcation point and then the Jacobian determinant does not change sign, so the determinant test is not conclusive. Thus tracking the determinant provides some guidance in finding bifurcation points, but it is not foolproof.

Bifurcation points can also be estimated by tracking the eigenvalues of the system. The estimation of eigenvalues is simplified in solid mechanics because the stress appears linearly in the geometric stiffness and varies approximately in proportion with the load. As we have seen from Example 6.?, the stability of an equilibrium path chamges when the lowest eigenvalue of the system changes sign. Thus the critical points can be located by an eigenvalue problem. There are several ways to do this:

1. interpolate the Jacobian matrix of the system by a linear relationship.
2. assume that the geometric and load stiffness are linearly proportional to the load parameter in the neighborhood under consideration;

Both methods can be applied with only a single load parameter. In the first method, we assume that the Jacobian A, is a linear function of the load parameter $\gamma$. The Jacobian matrix in the vicinity of the state $n$ can then be written in terms of the states around $n-1$ and $n$ by

$$
\begin{align*}
& \mathbf{A}(\mathbf{d}, \gamma)=(1-\xi) \mathbf{A}\left(\mathbf{d}_{n-1}, \gamma_{n-1}\right)+\xi \mathbf{A}\left(\mathbf{d}_{n}, \gamma_{n}\right) \equiv(1-\xi) \mathbf{A}^{-}+\xi \mathbf{A}^{0}  \tag{6.5.21}\\
& \gamma=(1-\xi) \gamma_{n-1}+\xi \gamma_{n} \tag{6.5.21b}
\end{align*}
$$

where the last term in the above defines a more compact notation we will use in the following. At the critical point, the determinant of the Jacobian matrix $\mathbf{A}$ vanishes, so

$$
\begin{equation*}
\operatorname{det} \mathbf{A}\left(\mathbf{d}, \boldsymbol{\gamma}_{c r i t}\right)=0 \tag{6.5.22}
\end{equation*}
$$

From (6.5.21) and the fact that a ystem with a zero determinant has a nontrivial homogenrous soloution, we deduce that there exists a $\xi$ such that

$$
\begin{equation*}
(1-\xi) \mathbf{A}^{-} \mathbf{y}+\xi \mathbf{A}^{0} \mathbf{y}=0 \tag{6.5.23}
\end{equation*}
$$

This can be put in the standard form of the generalized eigenvalue problem by the following rearrangement of terms:

$$
\begin{equation*}
\mathbf{A}^{0} \mathbf{y}=\xi\left(\mathbf{A}^{0}-\mathbf{A}^{-}\right) \mathbf{y} \tag{6.5.24}
\end{equation*}
$$

The solution of this eigenvalue problem then gives an estimate of the critical load by (6.5.21b).

The lowest eigenvalues of (6.5.24) can be either positive or negative. Negative eigenvalues indicate critical points which have been passed and are known about, or they may indicate critical values which have inadvertently been passed. In the latter case, state $n$ may no longer be on a stable equilibrium path.

For many structural problems, the eigenvalue problem may be simplified by taking advantage of the following:

1. the material stiffness in a linear material will not change significantly if the displacements prior to the critical point are small;
2. the geometric stiffness depends linearly on the load parameter, since it depends almost linearly on the stresses if the displacements are small (see the geometric stiffness for the bending and axial response in Eqs. ());
3. the load is independent of the displacements, so the load stiffness vanishes.

Since the geometric stiffness varies linearly with the load, if the above three conditions are met we can then write

$$
\begin{equation*}
\mathbf{A}^{0}=\mathbf{K}^{m a t}+\mathbf{K}^{g e o}\left(\lambda^{0}\right), \quad \mathbf{A}^{-}=\mathbf{K}^{m a t}+\mathbf{K}^{g e o}\left(\lambda^{-}\right) \tag{6.5.25}
\end{equation*}
$$

where $\mathbf{K}^{g e o}$ is the geometric stiffness for a unit value of the load parameter. Substituting into () then gives

$$
\begin{equation*}
\mathbf{K}^{m a t} \mathbf{y}=\xi\left(\mathbf{K}^{g e o}\left(\lambda^{0}\right)-\mathbf{K}^{g e o}\left(-\lambda^{-}\right)\right) \mathbf{y} \tag{6.5.26}
\end{equation*}
$$

The critical load is then given by

$$
\begin{equation*}
\lambda_{c r i t}=\xi\left(\lambda^{0}-\lambda^{-}\right) \tag{6.5.27}
\end{equation*}
$$

The procedure of determining the location of a nearly critical point then consists of storing the following and they geometric stiffness is saved from the last step, and using the current geometric and material stiffness, the eigenvalues are obtained. The eigenvalue which leads to the smallest critical load is the one of interest. When the parameter $0 \leq \xi \leq 1$, the critical point has been passed in the last step. When $\xi>1$, the critical point is estimated to be further ahead in the branch.

In analyzing structures, it is often desirable to estimate the first bifurcation point along the fundamental equilibrium path after a single load increment $\lambda_{1}$ has been applied. An initial estimate of the bifurcation point can be found in terms of the geometric stiffness computed after one load increment. We assume that $\mathbf{K}^{g e o}\left(\lambda^{-}\right)=\mathbf{0}$ since the first step is stress-free. Then Eq () gives

$$
\begin{equation*}
\mathbf{K}^{m a t} \mathbf{y}=\xi \mathbf{K}^{g e o}\left(\lambda^{0}\right) \mathbf{y} \tag{6.5.28}
\end{equation*}
$$

The critical value of the load parameter is then $\lambda_{\text {crit }}=\xi \lambda^{0}$, where $\lambda^{0}$ is the load parameter for the first load increment. this is the formula commonly cited in matrix structural texts for obtaining the buckling load of a structure. Note that it assumes that the geometry of the structure changes so little with increasing load that the first estimate of the geometric stiffness suffices for extrapolating the critical point. It is effective primarily for bifurcation points. Prior to reaching a limit point, the geometric stiffness changes significantly, so an estimate based on () is quite erroneous.

The study of systems stability has in the past two decades become a rich science known as dynamical systems theory. It includes topics such as chaos, fractals, attractors, repellors. These topics are beyond the scope of this book; some good references are Argyris and Melenk (), Moon () and Temam ().

Example 6.4. A simple example of a problem with stable and unstable paths connected by a turning point is the shallow truss shown in Fig. 6.11. The initial cross sectional areas of the elements are $A_{0}$ and the initial lengths of the two elements are $\ell_{0}$, which is given by $\ell_{0}^{2}=a^{2}+b^{2}$. A vertical load $f$ is applied as shown, and since this is the only load we consider $f$ to be the load parameter. The material is governed by a Kirchhoff law, (see Eq. ())

$$
\begin{equation*}
S=C E_{x} \tag{E6.1.1}
\end{equation*}
$$

where $C$; as pointed out in Section ??, for a small strain, large-displacement problem such as this, this constitutive equation is almost to a small-strain elastic, linear law. We will determine the equilibrium path as a function of the load and determine which branches are stable.

The deformation of the truss in is described by the variable $y$, the current vertical coordinate of the centerpoint, which leads to simpler equations than using the displacement. Since this material is path-independent, we can use the theorem of minimum potential energy to develop the discrete equations. The potential energy, Eq.(), in this case is given by

$$
\begin{equation*}
W=W^{i n t}-W^{e x t}, \quad W^{i n t}=\frac{1}{2} \sum_{e=1}^{2} \int_{\Omega_{0}^{e}} C \hat{E}_{x x}^{2} d \Omega, W^{e x t}=f(b-y) \tag{E6.1.2}
\end{equation*}
$$

where the Green strain is uniaxial with a only component along the axes of the bars contributing to the internal energy. The Green strain in for both elements is most easily evaluated by Eq.(), which gives

$$
\begin{equation*}
\hat{E}_{x x}=\frac{1}{2}\left(\ell^{2}-\ell_{0}^{2}\right)=\frac{1}{2}\left(a^{2}+y^{2}-a^{2}-b^{2}\right)=\frac{1}{2}\left(y^{2}-b^{2}\right) \tag{E6.1.3}
\end{equation*}
$$

so the internal energy is given by

$$
\begin{equation*}
W^{i n t}=k\left(y^{2}-b^{2}\right)^{2} \quad \text { where } k=\frac{1}{4} C A_{0} \ell_{0} \tag{E6.1.4}
\end{equation*}
$$

Combining the above with the potential of the external forces gives the total potential

$$
\begin{equation*}
W=k\left(y^{2}-b^{2}\right)^{2}-f(b-y) \tag{E6.1.5}
\end{equation*}
$$

The equilibrium equation is now obtained by applying the theorem of minimum potential energy, which states that the equilibrium equation is given by the stationary points of the potential W given above, so the equilibrium equation is

$$
\begin{equation*}
0=\frac{d W}{d y}=4 k\left(y^{2}-b^{2}\right) y+f \tag{E6.1.6}
\end{equation*}
$$

As can be seen from the above, the force is a cubic function of the vertical position of the centerpoint, which is shown in Fig. ?. The equilibrium path has two turning points, usually called limit points in structural mechanics, and three branches, denoted by $\mathrm{AB}, \mathrm{BC}$ and CD in Fig. ?.

We will now examine the stability of the branches of the equilibrium path. The dynamic response is examined at a position $y_{0}$ subject to a perturbation. A solution to the linearized equations is considered, so

$$
\begin{equation*}
y=y_{0}+\bar{y} \text { where } \bar{y}=\varepsilon e^{\mu t} \tag{E6.1.7}
\end{equation*}
$$

where $\varepsilon$ is a small parameter. The equations of motion for this problem are given by

$$
\begin{equation*}
M \frac{d^{2} y}{d t^{2}}=f^{e x t}-f^{i n t}=f_{0}-4 k\left(y^{2}-b^{2}\right) y \tag{E6.1.8}
\end{equation*}
$$

where $M$ is the mass of the node. Substituting () into () gives

$$
\begin{equation*}
f_{0}-4 k\left(\left(y_{0}+\bar{y}\right)^{2}-b^{2}\right)\left(y_{0}+\bar{y}\right)=M \frac{d^{2} \bar{y}}{d t^{2}} \tag{E6.1.9}
\end{equation*}
$$

Expanding the above and dropping all terms which are higher order than linear in $\bar{y}$, gives (it is expressed in terms of

$$
\begin{equation*}
f_{0}-4 k\left[y_{0}\left(y_{0}^{2}-b^{2}\right)+\bar{y}\left(3 y_{0}^{2}-b^{2}\right)\right]=M \frac{d^{2} \bar{y}}{d t^{2}} \tag{E6.1.10}
\end{equation*}
$$

The load cancels the first term in the brackets, so the equations of motion become

$$
\begin{equation*}
M \frac{d^{2} \bar{y}}{d t^{2}}+4 k \bar{y}\left(3 y_{0}^{2}-b^{2}\right)=0 \rightarrow \alpha= \pm i\left(3 y_{0}^{2}-b^{2}\right)^{\frac{1}{2}} \tag{E6.1.11}
\end{equation*}
$$

where the (b)follows from substituting () into (a). It can then be seen that the perturbation solution () is real with one positive exponential whenever $3 y_{0}^{2}-b^{2}<0$. So the branch defined by

$$
\begin{equation*}
-b / \sqrt{3}<y_{0}<b / \sqrt{3} \text { isunstable } \tag{E6.1.12}
\end{equation*}
$$

The results of the above stability analysis can be obtained directly be examining the second derivative of the potential energy function, which from () is given by

$$
\begin{equation*}
\frac{d^{2} W}{d y^{2}}=4 k\left(3 y^{2}-b^{2}\right), \frac{d^{2} W}{d y^{2}}<0 \text { when }-b<\sqrt{3} y<b, \frac{d^{2} W}{d y^{2}}>0 \text { otherwise } \tag{E6.1.13}
\end{equation*}
$$

So in the unstable equilibrium branches of a conservative system, the second derivative of the potential energy changes sign. The results of () and () are in fact identical: for a system which has no velocity dependent terms, the perturbation analysis is identical to taking a derivative of the forces, and the stability of the result of the perturbation analysis simply depends on the sign of the derivative of the forces, which is the second derivative of the work potential.

The linearized test for stability used in Example 8.?.1 is not a foolproof test for stability. For example, if $y_{0}=-0.99 b$, the test indicates that the equilibrium point is unstable. However, a numerical dynamics solution when started at tat point, will only oscillate with an amplitude of 0.002 , which to most engineers would not be an instability. The test as posed in Example 1 checks whether any perturbation will grow at all and conform to the criterion () based on a linearized analysis, which need not conform to any physical notion of instability. In contrast to the exponential instabilities seen in stability analysis of numerical methods, the instabilities in physical systems will not exhibit continuing growth. What it is does predict accurately is that when the dynamics is added to the system, the system will not oscillate about the unstable equilibrium point in response to a perturbation but move to oscillating about a nearby point on a stable equilibrium path.

Example 6.5. Consider a linear stability analysis of the beam element shown in Fig. 6.5E. Node 2 is clamped, node 1 is free to rotate and move in the $x$ direction. Find the equilibrium equation and the equilibrium branches of the system.


Figure 6.5E. Beam model used for stability analysis and equilibrium paths.
The displacement boundary conditions imply that

$$
\begin{equation*}
u_{x 1}=u_{y 1}=\theta_{1}=u_{y 2}=0 \tag{E6.5.1}
\end{equation*}
$$

Therefore, the only nonzero degrees-of-freedom are $u_{y 1} \equiv u_{1}$ and $\theta_{1}$. The equations of equilibrium can be deduced from Example ??? to be

$$
\begin{align*}
& \frac{E A}{\ell} u_{1}-\frac{2 E A}{15} \theta_{1}^{2}=F  \tag{E6.5.2}\\
& -\frac{2 E A}{15} \theta_{1} u_{1}+\left(\frac{4 E I}{\ell}-\frac{2 E A}{15} u_{1}+\frac{3 E A P}{35}\right) \theta_{1}=0 \tag{E6.5.3}
\end{align*}
$$

The above system of two nonlinear algebraic equations in two unknowns possesses two solutions:

$$
\begin{align*}
& \text { Solution 1: } \quad \theta_{1}=0, u_{1}=\frac{P \ell}{E A}  \tag{E6.5.4}\\
& \text { Solution 2: } u_{1}=\frac{2 \ell}{15} \theta_{1}^{2}+\frac{P \ell}{E A} \\
& u_{1}=\frac{P \ell}{28} \theta_{1}^{2}+\frac{15 I}{A \ell}
\end{align*}
$$

These two curves are plotted in Figure 6. It can be seen a pitchfork bifurcation occurs at

$$
\begin{equation*}
u_{1}=\frac{15 I}{A \ell} \tag{E6.5.7}
\end{equation*}
$$

This is the critical point for this beam. The corresponding load can be found by substituting (???) and $\theta_{1}=0$ into Eq. (E6.5.2), which gives

$$
\begin{equation*}
F_{c r i t}=\frac{15 E I}{\ell^{2}} \tag{E6.5.8}
\end{equation*}
$$

The linearized stability of any of the equilibrium paths can be examined by considering the linearized equations of motion about a point on the path:

$$
\begin{equation*}
\mathbf{M} \Delta \ddot{\mathbf{d}}+\left(\mathbf{K}_{\text {mat }}+\mathbf{K}_{g e o}\right) \Delta \mathbf{d}=0 \tag{E6.5.9}
\end{equation*}
$$

where $\Delta \mathbf{d}$ here is the displacement from the path. The equations can be written out by using the mass matrix given in Eq. (9.3.18) and the material and tangent stiffnesses given in Eqs. (???) and (???). The resulting equations are

$$
\frac{\rho_{0} \ell_{0} A_{0}}{420}\left[\begin{array}{cc}
210 & 0  \tag{E6.5.10}\\
0 & \alpha \ell^{2}
\end{array}\right]\left\{\begin{array}{l}
\Delta \ddot{u}_{1} \\
\Delta \ddot{\theta}_{1}
\end{array}\right\}+\left[\begin{array}{ll}
\frac{A E}{\ell}+ & \\
& \frac{4 E I}{\ell}
\end{array}\right]\left\{\begin{array}{l}
\Delta u_{1} \\
\Delta \theta_{1}
\end{array}\right\}=0
$$

We will examine the stability of two of the paths for $u_{1}>15 I / A \ell$; the path $P A$ and the path $P C$.

The problem parameters are Young's modulus E, the moment of the crosssection I, and the original length of the beam $\ell_{o}$. The beam is modeled by a single element with a linear axial displacement field and a cubic transverse displacement field. This is a standard beam element described in Chapter 9. The unknowns are $\mathbf{d}^{T}=\left[\begin{array}{lll}u_{x} & u_{y} & \theta\end{array}\right]$, where $\theta$ is the rotation of the node; nodal subscripts have been dropped because they all refer to node 1 .

## NUMERICAL STABILITY

At this point it is worthwhile to comment on the differences between physical stability and numerical stability. Physical stability pertains to the stability of an solution of a model, whereas numerical stability pertains to the stability of the numerical solution. Numerical instabilities arise from the discretization of the model equations, whereas physical instabilities are instabilities in the solutions of the model equations independent of the numerical discretization. Numerical stability is usually only examined for processes which are physically stable. Very little is known how a "stable" numerical procedures behave in physically unstable processes. This shortcoming has important practical ramifications, because many computations today simulate physical instabilities, and if we cannot guarantee that our methods track these instabilities accurately, then these simulations may be suspect.

Numerical stability of a time integration procedure is defined in analogously to stability of solutions, Eq. ( 6.5.1-2). A numerical procedure is stable if small perturbations of initial data result in small changes in the numerical response. More formally, the numerical procedure is stable if

$$
\begin{equation*}
\left|\mathbf{u}_{A}^{n}-\mathbf{u}_{B}^{n}\right| \leq C \varepsilon \quad \forall n>0 \tag{6.5.29}
\end{equation*}
$$

when

$$
\begin{equation*}
\left|\mathbf{u}_{A}^{0}-\mathbf{u}_{B}^{0}\right| \leq \varepsilon \tag{6.5.30}
\end{equation*}
$$

LATERIt is of interest to note that numerical stability of a process that is physically unstable cannot be examined by this definition, i.e. we cannot say anything about the stability of a numerical procedure when applied an equation that exhibits unstable response. The reason can be seen as follows. If a system is unstable, then the solution to the system will not satisfy (). Therefore, even if the numerical solution procedure is stable, it will not satisfy ().

General results for numerical stability of time integrators are largely based on the analysis of linear systems. These results are extrapolated to nonlinear systems by applying them to the linearized equations. Therefore, we will first describe the stability theory which is used to obtain critical time steps for linear systems. Next we described the procedures for applying these results to nonlinear systems. In conclusion, we will describe some results on stability of time integrators which apply directly to nonlinear systems. However, we stress that at the present time there is no stability theory which encompasses the nonlinear problems which are routinely solved by nonlinear finite element methods, and most of our insight into stability stems from the analysis of linear models.

Numerical Stability of Linear Systems. Most of the theory of stability of numerical methods is concerned with linear systems. The idea is that if a numerical method is unstable for linear systems, it will of course be unstable for nonlinear systems also, since linear systems are a subset of nonlinear systems. Luckily, the converse has also turned out to be true: numerical methods which are stable for linear systems in almost all cases turn out to be stable for nonlinear systems. Therefore, the stability of numerical procedures for linear systems provides a useful guide to their behavior in both linear and nonlinear systems.

To begin our exploration of stability of numerical procedures, and in particular the stability of time integrators, we first consider the equations of heat conduction:

$$
\begin{equation*}
\mathbf{M \dot { u }}+\mathbf{K} \mathbf{u}=\mathbf{f} \tag{6.5.31}
\end{equation*}
$$

where M is the capacitance matrix, K is the conductance matrix, f is the forcing term and $u$ is a matrix of nodal temperatures. This system is chosen as a starting point because it is a first order system of ordinary differential equations, while the equations of motion are second order in time.

To apply the definition of stability, we consider two solutions for the same system with the same discrete forcing function but slightly different initial data. The two solutions satisfy the same equation with the same $\mathbf{f}$, so

$$
\begin{equation*}
\mathbf{M} \dot{\mathbf{u}}_{A}+\mathbf{K} \mathbf{u}_{A}=\mathbf{f} \quad \mathbf{M} \dot{\mathbf{u}}_{B}+\mathbf{K} \mathbf{u}_{B}=\mathbf{f} \tag{6.5.32}
\end{equation*}
$$

If we now take the difference of the two equations, we obtain

$$
\begin{equation*}
\mathbf{M d}+\mathbf{K d}=0 \quad \mathbf{d}=\mathbf{u}_{A}-\mathbf{u}_{B} \tag{6.5.33}
\end{equation*}
$$

We now consider a two-step family of time integrators:

$$
\begin{equation*}
\mathbf{d}_{n+1}=\mathbf{d}_{n}+(1-\alpha) \Delta t \dot{\mathbf{d}}_{n}+\alpha \Delta \dot{\mathbf{d}}_{n+1} \tag{6.5.34}
\end{equation*}
$$

Since (6.5.33) holds at time steps $n$ and $n+1$, we can multiply them respectively by $(1-\alpha) \Delta t$ and $\alpha \Delta t$, respectively

$$
\begin{equation*}
(1-\alpha) \Delta t \mathbf{M} \dot{\mathbf{d}}_{n}+(1-\alpha) \Delta t \mathbf{K} \mathbf{d}_{n}=0, \quad \alpha \Delta t \mathbf{M} \dot{\mathbf{d}}_{n+1}+\alpha \Delta t \mathbf{K} \mathbf{d}_{n+1}=0 \tag{6.5.35}
\end{equation*}
$$

Adding the above two equations and using (6.5.34) to eliminate the derivatives, we obtain

$$
\begin{equation*}
(\mathbf{M}+\alpha \Delta t \mathbf{K}) \mathbf{d}_{n+1}=(\mathbf{M}+(1-\alpha) \Delta t) \mathbf{K} \mathbf{d}_{n} \tag{6.5.36}
\end{equation*}
$$

This equation is in general amplification matrix form: it gives the numerical solution at times step $n+1$ in terms of the solution at time step $n$. An amplification matrix $\mathbf{A}$ is a matrix which gives the solution at time step $n+1$ in of the solution at time step $n$ by

$$
\begin{equation*}
\mathbf{d}_{n+1}=\mathbf{A} \mathbf{d}_{n} \tag{6.5.37}
\end{equation*}
$$

The generalized amplification matrix form is

$$
\begin{equation*}
\mathbf{B} \mathbf{d}_{n+1}=\mathbf{A} \mathbf{d}_{n} \tag{6.5.38}
\end{equation*}
$$

We shall now show that the time integrator is stable if the eigenvalues of the generalized amplification matrix form lie within the unit circle in the complex plane.

For this purpose, we need to recall the eigenvalue problem associated with (6.5.33):

$$
\begin{equation*}
\mathbf{K} \mathbf{y}_{i}=\lambda_{i} \mathbf{M} \mathbf{y}_{i} \tag{6.5.39}
\end{equation*}
$$

where $\lambda_{i}$ are the eigenvalues and $\mathbf{y}_{i}$ the eigenvectors of the system. We recall that the matrix $\mathbf{M}$ is positive definite and symmetric, whereas the matrix $\mathbf{K}$ is positive semidefinite and symmetric. Because of the symmetry of the matrices, the eigenvectors of (6.5.39) are orthogonal with respect to $\mathbf{M}$ and $\mathbf{K}$, which can be written as

$$
\begin{equation*}
\mathbf{y}_{j} \mathbf{M} \mathbf{y}_{i}=\delta_{i j}, \quad \mathbf{y}_{j} \mathbf{K} \mathbf{y}_{i}=\lambda_{i} \delta_{i j}(\text { nosumon } i) \tag{6.5.40}
\end{equation*}
$$

and from the positiveness of the matrices the eigenvalues are nonnegative. The generalized amplification equation is associated the generalized eigenvalue problem

$$
\begin{equation*}
\mathbf{A} \mathbf{z}_{i}=\mu_{i} \mathbf{B} \mathbf{z}_{i} \tag{6.5.41}
\end{equation*}
$$

The eigenvalues of the above system will be shown to control the stability of the time integrator. In general, these eigenvalues may be complex. Stability then requires that the moduli of all of the eigenvalues be less or equal to 1 . Otherwise at least one component of the solution grows exponenetially like $z^{n}$, so the solution is unstable. In other words, if we consider the complex plane as shown in Fig, X, then the eigenvalues must lie within or on the unit circle for the numerical method to be stable.

The eigenvectors span the space $R^{n}$, so any vector $\mathbf{d} \in R^{n_{D}}$ can be written as a linear combination of the eigenvalues, see XXX,. The eigenvectors of (6.5.41) and are identical to the eigenvectors of the (6.5.39) and the eigenvalues are related by the following:

$$
\begin{equation*}
\text { if } \mathbf{A}=a_{1} \mathbf{M}+a_{2} \mathbf{K} \text { and } \mathbf{B}=b_{1} \mathbf{M}+b_{2} \mathbf{K} \text { then } \mu=\frac{a_{1}+a_{2} \lambda_{i}}{b_{1}+b_{2} \lambda_{i}} \tag{6.5.42}
\end{equation*}
$$

This is shown as follows. Since the eigenvectors $\mathbf{y}_{i}$ span the space, we can expand the eigenvectors $\mathbf{z}_{i}$ in terms of $\mathbf{y}_{i}$ by

$$
\begin{equation*}
\mathbf{z}_{i}=c_{i} \mathbf{y}_{i} \tag{6.5.43}
\end{equation*}
$$

Substituting the above into (6.5.41), premultiplying by $\mathbf{y}_{j}$ and using the orthogonality relations (6.5.40) gives

$$
\begin{equation*}
a_{1}+a_{2} \lambda_{i}=\mu_{i}\left(b_{1}+b_{2} \lambda_{i}\right) \tag{6.5.44}
\end{equation*}
$$

from which the last equation in (6.5.42) follows immediately.

We now ascertain the conditions under which the eigenvalues $\mu_{i}$ fall within the unit circle, which corresponds to a stable numerical integration. Using again the fact that the eigenvectors $\mathbf{y}_{i}$ span the space, expand the initial solution vector at $t=0$ in terms of the eigenvectors by

$$
\begin{equation*}
\mathbf{d}_{0}=\sum_{i=1}^{n_{D}} r_{0}^{i} \mathbf{y}_{i} \tag{6.5.45}
\end{equation*}
$$

where $r_{0}^{i}$ is determined by the initial conditions. Substituting the above into () and using the fact that $\mathbf{y}_{i}$ are also eigenvectors of () with eigenvalues $\mu_{i}$, we obtain that

$$
\begin{equation*}
\mathbf{d}_{1}=\sum_{i=1}^{n_{D}} \mu_{i} r_{0}^{i} \mathbf{y}_{i}, \quad \mathbf{d}_{2}=\sum_{i=1}^{n_{D}}\left(\mu_{i}\right)^{2} r_{0}^{i} \mathbf{y}_{i}, \quad \mathbf{d}_{n}=\sum_{i=1}^{n_{D}}\left(\mu_{i}\right)^{n} r_{0}^{i} \mathbf{y}_{i} \tag{6.5.46}
\end{equation*}
$$

where the second equation follows by repeating the process and the last equation can be obtained by induction. We can see immediately from the above that if any of the eigenvalues of the generalized amplification matrix $\mu_{i}$ is greater than one, the solution will grow exponentially. Since we are examining the behavior of the difference of two solutions, this indicates that the procedure is unstable. Although some readers will advance the counterargument that this unstable growth will occur only if the initial data contains the eigenvector associated with $\mu_{i}$, in fact, due to roundoff error, the constant $r_{i}^{0}$ will be initially be nonzero or become nonzero later in the calculation. No matter how small the constant, the exponenetial growth will dominate ina very few time steps.

Using Eqs. (6.5.42) and (6.5.36) it follows that

$$
\begin{equation*}
\mu_{i}=\frac{1-\alpha \Delta \lambda \lambda_{i}}{1+\alpha \Delta \lambda \lambda_{i}} \tag{6.5.47}
\end{equation*}
$$

Since this eigenvalue is always real, the stability condition can be written as $\left|\mu_{i}\right| \leq 1$. We consider eigenvalues $\left|\mu_{i}\right|=1$ to lead to stable solutions at this point, but this is not always the case. From the preceding we deduce the conditions on the time step necessary for numerical stability as follows:

$$
\begin{align*}
& \mu_{i} \leq 1 \rightarrow \frac{1-(1-\alpha) \Delta t \lambda_{i}}{1-\alpha \Delta \lambda_{i}} \leq 1 \rightarrow \text { always met }  \tag{6.5.48}\\
& \mu_{i} \geq-1 \rightarrow \frac{1-(1-\alpha) \Delta t \lambda_{i}}{1-\alpha \Delta t \lambda_{i}} \geq-1 \rightarrow(1-2 \alpha) \Delta t \lambda_{i} \leq 2 \tag{6.5.49}
\end{align*}
$$

There are two distinct consequences of Eq.(). If $1-2 \alpha \geq 0$, i.e. $\alpha \geq 0.5$, then the condition of stability is met regardless of the size of the time step. The method is
then called unconditionally stable. When $1-2 \alpha<0$, i.e. $\alpha<0.5$, Eq (6.5.49) yields the requirement that

$$
\begin{equation*}
\Delta t \leq \frac{2}{(1-2 \alpha) \lambda_{i}} \forall i \tag{6.5.50}
\end{equation*}
$$

where we have indicated that the condition on the eigenvalue $\mu_{i}$ must be met for all $i$. The maximum eigenvalue then sets the time step, so the critical time step is given by

$$
\begin{equation*}
\Delta t \leq \max _{i} \frac{2}{(1-2 \alpha) \lambda_{i}} \text { or } \Delta t_{c r i t}=\frac{2}{(1-2 \alpha) \lambda_{\max }} \tag{6.5.51}
\end{equation*}
$$

A method which is stable only for time steps below a critical value is called conditionally stable. If we consider the explicit form of this generalized update equation, i.e. $\alpha=0$, then the above gives

$$
\begin{equation*}
\Delta t_{c r i t}=\frac{2}{\lambda_{\max }} \tag{6.5.52}
\end{equation*}
$$

Thus the stable time step is inversely proportional to the maximum eigenvalue of the system. The stiffer the system, the smaller the stable time step. For the trapezoidal rule, $\alpha=05$, and for any $0.5<\alpha \leq 1$ the method is unconditionally stable. For $0 \leq \alpha<0.5$, the integrator is implicit but conditionally stable, so these values of $\alpha$ are of little practical value.

To give the reader a appreciation of the explosive growth of an exponential instability, Table ? shows the results for exponential growth for several values of the eigenvalue $\mu_{i}$. Exponential growth is truly startling. It is also the reason why compound interest can make you very rich if you live long enough and start saving early.

In summary, we have shown that the determination of the stability of an integration formula for the semidiscrete initial value problem () can be reduced to examining the eigenvalues of the generalized amplification matrix (). If any eigenvalue lies outside the unit circle in the complex plane, the perturbation grows exponentially so the solution is numerically unstable. Otherwise, the method is stable.

Stability of thhe Central Difference Method. We now use the same techniques to examine the stability of the central difference method for the equations of motion.

## MATERIAL STABILITY

An important issue in modern computational mechanics is the stability of the material models. The issue has already been discussed on several occasions in Chapter 5, cf...In this Section, we examine the implications of material instability on computational procedures and provide some remedies for the major difficulties.

As pointed out in Chapter 5, material instability results from the loss of positive definiteness in the tangent modulus tensor relating the Truesdell rate of the Cauchy stress to the rate of deformation. The name material instability is a slight misnomer because the occurrence of this phenomenon does not lead automatically to the violation of stability definitions such as (6.5.1). Instead, an unstable material is characterized by the possibility of unbounded spectral growth for a body in a homogeneous state of stress. When a material fails to meet the stability criteria for a subdomain of the problem, unbounded growth of the solution does not necessarily occur.

Nevertheless, the consequences in a computation of the failure to meet material stability criteria are dramatic: for rate-independent materials, loss of material stability changes the PDE locally from hyperbolic to elliptic in dynamic problems and vice versa in static problems. Furthermore, in rate indenpendent materials this is accompanied by a phenomenon called localization to a set of measure zero: the domain in which material instability occurs in a three dimensional problem will localize to a surface. On that surface in the domain, the strains will be infinite and the motion will be discontinuous. Although this ostensibly looks like a good way to model fracture and failure of materials, because of the localization to a set of measure zero, the dissipation associated with this process vanishes, so that the model is inappropriate for any realistic physical model of fracture or shear banding.

The literature on material instability goes back at least as far as Hadamard (1906). I haven't read the literature of that time, and even my knowledge of Hadamard is second-hand, so there could be earlier studies. Hadamard examined the question of what happens when the tangent modulus in a small deformation problem is negative. He concluded that according to the wave equation and the formula for the wavespeed, (???), that the wavespeed is then imaginary (the square root of a negative number), so such materials could not exist.

The next major milestone in the study of unstable materials is the work of Hill (??), who examined the conditions under which materials are unstable. His methodology was to consider the momentum equation for a homogeneous state of initial stress in terms of the displacements. The momentum equation is then

$$
C_{i j k l} v_{k, l}=\rho \ddot{v}_{i} \text { wrong eqn unless } \mathrm{v}=\mathrm{displ}
$$

Using the technique of linear stability analysis, he examined the growth and decay of solutions of the form

$$
u_{i}=A_{i} e^{\mathrm{k}(x-c t)}
$$

The solution grows exponentially if any of the eigenvalues of the problem

$$
u_{i}=A_{i} e^{\mathrm{k}(x-c t)}
$$

are negative. He also showed that equivalently one could examine the material instability through the possibility of acceleration waves. This technique is now classical and is used in finite elements to detect the possibility of material instability> It goes as follows:

Hill() also examined material instabilities for large deformation problems and the question of which rate is appropriate for ascertaining unstable behavior. he concluded that

Another milestone paper in this stream is the work of Rudnicki and Rice(??), who showed that material instabilities can occur even in the presence of strain hardening when the plasticity is nonassociative. The argument has been given in Section 5.?

Thus when computers came on the scene for nonlinear analysis in the 1970's there were two known causes of material instability: a negative modulus (or a negative eigenvalue of the tangent modulus matrix) and a nonassociative plasticity law. Computational analysts soon began to include material models which included either or both of these and they discovered many difficulties. In fact it was argued by many, including Drucker and Sandler(), that material models that violate the stability postulates should never be used in computational methods. Their arguments proved fruitless since there is no way to replicate observed phenomena such as shear banding without a model that exhibits strain softening, although the models which were first used to examine shear bands, Clifton and Milliner(), are viscoplastic and satisfy the stability postulates.

Zdenek Bazant and I started studying the problem in 197? and based on some computational results of Hyun we surmised that the closed form solution for a rate-independent material model must exhibit an infinite strain. We were able to construct a one-dimensional solution of this behavior, albeit quite inelegant in retrospect, and learned that for these materials the unstable behavio must localize to a set of measure zero and that the dissipation would then vanish.

This led to the search for a regularization of the governing equations, which we called a localization limiter at the time. We soon discovered that both gradient models and nonlocal models regularize the solution, Bazant, Chang and Belytschko and Lasry and Belytschko(). This solution of remedying the difficulties associated with negative moduli had already occurred in another context, the heat equation, where Kahn and Hilliard() circumvented the difficulty by a gradient theory, which came to be known as the Kahn-Hilliard theory. Hilliard was incidentally also at Northwestern but we were unaware of his work until later. Aifantis(??) had proposed gradient regularization in solid mechaincs before us.

Subsequently a plethora of work emerged in this area, with two goals: to obtain physical ustifications for the regularization procedure and to simplify the treatment of nonlocal and gradient models. Schreyer et al (), introduced gradient theories based on the gradient of the plasticity parameter lambda in Eq.(5.??), Pijaudier-Cabot and Bazant(??) introduced the gradient on the damage parameter. These are important because introducing nonlocality in the 6 strain components is awkward indeed. Mulhaus and Vardoulakis showed that a coupled stress theory also regularizes the equations, and Needleman showed that viscoplasticity regularizes the equations. an important recent work is Triantifyllides and ?, who proposed a technique for relating unit cell models to the parameters in a nonlocal theory. deBorst et al (??) further investigated the Schreyer et al approach and showed that that consistency (5.??) requirement then intdroduces another partial differential equation into the system; the boundary conditions for these partial differential equations are still an enigma. Hutchinson and Fleck() showed
expreimentally that metal plasticity depends on scale and developed a gradient plasticity theory motivated by dislocation movement.

Regularization Techniques. There are thus four regularization techniques that are under study for unstable materials:

1. gradient regularization, in which a gradient of a field variable is introduced in the constitutive equation
2. integral, or nonlocal, regularization, in which the the constitutive equation is a function of a nonlocal variable, such as nonlocal damage, a nonlocal invariant of a strain, or a nonlocal strain.
3. coupled stress regularizaztion

4 , regularization by introducing time dependence into the material
All of these are except the last are still in an embryonic state of development. Little is known about the material constants and the associated material length scales which are required.

Regularization by introducing time dependence has progressed faster than the others because viscoplastic material laws has achieved a stat e of maturity by the time that localization became a hot area of research. However, viscoplastic regularization has some notable peculiarites: there is no constant length scale in the viscoplastic maodel and the solution in the presence of matrial instability is characterized by exponential growth. Therefore, although a discontinuity does not develop in te displacement as in the rate-independent strain-softening material, the gradient in thhe displacement increases unboundedly with time. Wright and Walter have shown that this anomaly can be rectified by coupling the momentum equation to heat conduction via the energy conservation equation. the length scales then computed agree well with observed shear band widths in metals.

The computational meodeling of localization still poses substantial difficulties. for most materials, the length scales of shear bands are much smaller than those of the body. Therefore tremendous resolution is required to obtain a reasonably accdurate solution to these problems, see Belytschko et al for some high resolution computations. Solutions converge very slowly with mesh refinement. This behavior of numerical solutions is often called mesh sensitivity or lack of objectivity, though it has nothing to do with objectivity or its absence: it is simply a consequence of the inabiloity of coarse meshes to resolves high gradient in viscopladtic materials or discontinuites in rate-independent solutions.

Several techniques have evolved to improve the coarse-mesh accuracy of finite element models for unstable materials. The first of these involve the embedment of discontinuities in the element. Ortiz ewt al were the first to do this: theyembedded discontinuites in the strain field of the 4 -node quadrilateral when the acoustic trensor indicated a material instability in the element. Belytschko, Fish and Engleman attempted to embed a displacement discontinuity by enriching the strain field with a narrow band where the unstable material behavior occurs. In the band, the material behavior was considered homogeneous, which is ridiculous since an unstable material cannot remain ina homogenous state of stress: any perturbation will trigger a growth on the scale of the perturbation. Such is hindsight. Nevertheless these models were able to capture the evolving discontinuity in displacement more effectively. Sime and ??? invoked the theory oof distributions to justify such techniques. They also categorized discontinuities as strong (in the displacements) and weak (in the strains). This categorization
incidentally is at odds with the widely used categorization in shocks in fluid dynamics, where discontinuites in occur in the velocity and the motion is continuous, see Section ??. These techniques have recently been further explored by Armero et al () and Garipakti ad Hughes (??).

Shear bands are closely related to fracture: a shear band can be viewed as a discontinuity in the tangential displacement, a fracture as a discopntinuity in all components of the displacement, see Chapter 3, Example ??. Just as shear bands can be viewed as the outcome of a material instability in the shear component, the development of a fracture can be viewed numerically as the outcome of a material instability in the directions normal (and tangential in the case of mode 2 fracture)to the discontinuity. The relationship of damage and fracture has long been noted, see LeMaitre and Chab oche (??), where a fracture is assumed to occur when the damage variable reaches 0.7 . the origin of the number 0,7 is quite hazy in most works on damage mechanics, but it can be seen to arise from the phase transition point based on percolation theory is 0.59275 , Taylor and Francis (1985). The modeling of fracture by dmage poses some of the same difficulties encountered in shear band modeling, since the material law becomesunstable when the damage excdeeds a threshold value. All of the phenomena found in shear banding then occur: localization to a set of measure zero for rateindependent models, exponential growth for simple rate-dependent models, zero dissipation in failure and absence of a length scale.

These difficulties were grasped and resolved in a novel way early in the evolution of fintie elements by Hillerborg et al (??), Basant (??) and Willam(??) have also contributed to this approach. The idea is to match the energy of fracture to the energy dissipated by the element in which the localization occurs.
[??] H.M. Hiller, T.J.R. Hughes, and R.L. Taylor, "Improved Numerical Dissipation for Time Integration Algorithms in Structural Dynamics," Earthquake Engineering and Structural Dyanmics, Vol. 5, 282-292, 1977.

The tangent moduli are denoted by $\mathcal{C}^{S E}$ and a general constitutive equation can be written as

$$
\begin{aligned}
& \dot{\mathbf{S}}=C^{S E}: \dot{\mathbf{E}} \text { or } \dot{S}_{i r}=C_{i r k l}^{S E} \dot{E}_{k l} \\
& P_{i j}=C_{i r k l} \dot{E}_{k l} F_{r j}^{T}+S_{i r} \dot{F}_{r j}^{T}
\end{aligned}
$$

Now using (3.3.20) to express $\dot{\mathbf{E}}$ in terms of $\dot{\mathbf{F}}$ and noting the minor symmetry of the tangent modulus marix (see Section 5.?) gives

$$
P_{i j}=C_{i r k l} F_{k m} \dot{F}_{l m} F_{r j}^{T}+S_{i r} \dot{F}_{r j}^{T}
$$

## Norms.

Norms are used in this book primarily for simplifying the notation. No proofs are given that rely on the properties of normed spaces so the student need only learn the definitions of the norms as given below. It is also worthwhile to learn an interpretation of a norm as a distance. This is easily grasped by first learning the norms in the space $\ell_{n}$, which is a norm in the space of vectors of real numbers. The extension to function spaces such as the Hilbert spaces and the space of Lebesque integrable functions, $\mathcal{L}_{2}$, (often named el-two) is then straightforward.

The norms on $\ell_{n}$ are defined by the following. We begin with the norm $\ell_{2}$, which is simply Euclidan distance. If we consider an $n$-dimensional vector a, often written as $\mathbf{a} \in R_{n}$, then the $\ell_{2}$ norm is given by

$$
\|\mathbf{a}\|_{2}=\left(\sum_{i=1}^{n} a_{i}^{2}\right)^{\frac{1}{2}}
$$

In the above, the symbol $\|\cdot\|$ indicates a norm and the subscript 2 in combination with the fact that the enclosed variable is a vector indicates that we are referring to the $\ell_{2}$ norm. For $n=2$ or 3 , respectively, the $\ell_{2}$ norm is simply the length of the enclosed vector. The distance between two points, or the difference between two vectors, is written as

$$
\|\mathbf{a}-\mathbf{b}\|_{2}=\left(\sum_{i=1}^{n}\left(a_{i}-b_{i}\right)^{2}\right)^{\frac{1}{2}}
$$

Fundamental properties of the $\ell_{2}$ norm are that:

1. it is positive,
2. it satisfies the triangle inequality
3. it is linear

The $\ell_{k}$ norms are generalizations for the above definition to arbitrary $k>1$ as follows:

$$
\|\mathbf{a}\|_{k}=\left(\sum_{i=1}^{n}\left|a_{i}\right|^{k}\right)^{\frac{1}{k}}
$$

Norms for $k \neq 2$ are seldom used except for $k=\infty$, which is called the infinity norm. The infinity norm gives the component of the vector with the maximum absolute value, which can easily be figured out by thinking about (??) a little bit. Thus we can write that

$$
\|\mathbf{a}\|_{\infty}=\max _{i}\left|a_{i}\right|
$$

One of the principal applications of these norms is to define the error in a vector. Thus if we have a approximate solution to a set of discrete equations $\mathbf{d}^{a p p}$ and the exact solution is $\mathbf{d}^{\text {exact }}$, then a measure of the error is

$$
\text { error }=\left\|\mathbf{d}^{a p p}-\mathbf{d}^{\text {exact }}\right\|_{2}
$$

If you are concerned with the maximum error in any component of the solution, then you should select the infinity norm. When the concern is with the error over a selected number of components, then the norm can be restricted to those components. The idea is that you use norms to achieve what you need: they are not immutable. In using norms to asses errors in solutions, it is recommended that the error be normalized, e.g.

because absolute errors are very difficult to interprete and are meaningless unless the approximate magnitude of the solution is reported.

Norms of functions are defined analogously to the above. The relationship between functions and vectors is that a function can be thought of as an infinte dimensional vector. Thus the norm in function space that corresponds to $\ell_{2}$ is given by

$$
\|a(x)\|_{\mathcal{L}_{2}}=\left(\sum_{i=1}^{n} a^{2}\left(x_{i}\right) \Delta x\right)^{\frac{1}{2}}=\left(\int_{0}^{1} a^{2}(x) d x\right)^{\frac{1}{2}}
$$

This norm is called the $\mathcal{L}_{2}$, and the space of functions for which this norm is well-defined and bounded is called the $\mathcal{L}_{2}$ space; usually just the number is indicated This space is the set of all functions which are square integrable, and it includes the space of all functions which are piecewise continuous.

The Dirac delta function $\delta(x-y)$ is defined by

$$
f(x)=\int_{-\infty}^{+\infty} f(y) \delta(x-y) d y
$$

is not square integrable. It can be thought of as a function which is infinite at $x=y$ but vanishes everywhere else. The mathematical definition of this function is the topic of the theory of Schwartz distributions, which is needed for a good understanding of convergence theory but not for nonlinear finite element analysis.

The exact delineation of the space $\mathcal{L}_{2}$ can get quite technical, since mathematicians are concerned with questions such as whether the function $f(x)=1$ when $x$ is rational, $f(x)=0$ otherwise, is square integrable (it is not).
But for engineers concerned with the finite element method, it is sufficient to know that any function mentioned in this book except the Dirac delta function posseses an $\mathcal{L}_{2}$ norm.

The space of functions $\mathcal{L}_{2}$ is a special case of a more general group of spaces called Hilbert spaces. The norm in the Hilbert space $\mathcal{H}_{1}$ is defined by

$$
\|a(x)\|_{\mathscr{H}_{1}}=\left(\int_{0}^{1}\left(a^{2}(x)+a_{, x}^{2}(x)\right) d x\right)^{\frac{1}{2}}
$$

Just as for vector norms, the major utility of these norms is in measuring errors in functions. Thus if the finite element solution for the displacement in a one dimensional problem is denoted by $u^{h}(x)$ and the exact solution is $u(x)$, then the error in the displacement can be measured by

$$
\text { error }=\left\|u^{h}(x)-u(x)\right\|_{\mathcal{L}_{2}}
$$

The error in the strain, i.e. the first derivative of the displacement, can be measured by the $\mathcal{H}_{1}$ norm. While this norm also includes the error in the function itself, the error in the derivative almost always dominates. On the other hand, you could measure the error in the strain by the $\mathcal{L}_{2}$ norm of the first
derivative. This is not a valid norm in mathematics, because it can vanish for a nonzero function (just take a constant), so it is called a seminorm.

These norms can be generalized to arbitrary domains in multi-dimensional space and to vector and tensors by just changing the integrals and integrands. Thus the
$\mathcal{L}_{2}$ norm of the displacement on a domain is given by

$$
\|\mathbf{u}(\mathbf{x})\|_{\mathcal{L}_{2}}=\left(\int_{\Omega} u_{i}(\mathbf{x}) u_{i}(\mathbf{x}) d \Omega\right)^{\frac{1}{2}}
$$

The definition of the $\mathcal{H}_{1}$ norm is somewhat more puzzling??? since as given in mathematical tests it is not a true scalar (it is not invariant with rotation):

$$
\|\mathbf{u}(\mathbf{x})\|_{\mathfrak{H}_{1}}=\left(\int_{\Omega} u_{i}(\mathbf{x}) u_{i}(\mathbf{x})+u_{i, j}(\mathbf{x}) u_{i, j}(\mathbf{x}) d \Omega\right)^{\frac{1}{2}}
$$

In general, the precise space to which a norm pertains is not given. Usually only a number, or even nothing is given by the norm sign. The norm must then be inferred from the context.

In linear stress analysis, the energy norm is often used to measure error. It is given by

$$
\text { energy norm }=\left(\int_{\Omega} \varepsilon_{i j}(\mathbf{x}) C_{i j k l} \varepsilon_{k l}(\mathbf{x}) d \Omega\right)^{\frac{1}{2}}
$$

Its behavior is similar to that of the $\mathcal{H}_{1}$ norm.

